Semidefinite Programming Relaxations of the Traveling Salesman Problem

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The Traveling Salesman Problem (TSP)

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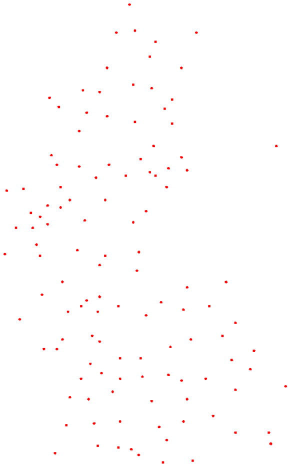
Given a set of cities, find the shortest *tour* that visits all cities and returns to the start.



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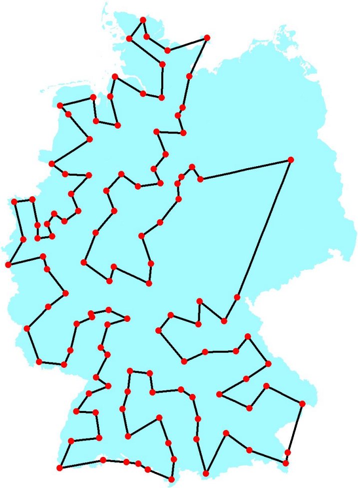
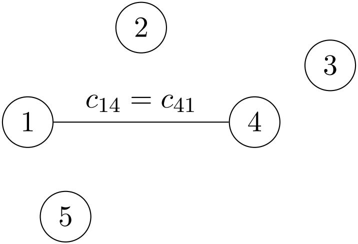
Images from
`www.math.uwaterloo.ca/tsp`



The (Symmetric, Metric) TSP

- Complete undirected graph K_n
- Edge costs c_{ij} for distinct $i, j \in [n] = \{1, 2, ..., n\}$ with $c_{ij} = c_{ji}$ and $c_{ij} \leq c_{ik} + c_{kj}$ for all distinct i, j, k



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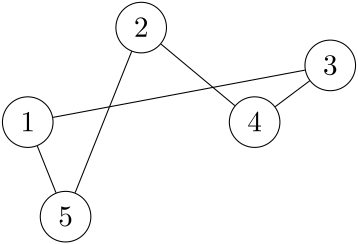

Goal

Find a minimum-cost Hamiltonian cycle: the cheapest cycle visiting every city exactly once.

Solving the TSP

"I conjecture that there is no good [polynomial-time] algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know."

– Jack Edmonds (1967)

TSP is hard

Finding an optimal solution is known to be NP-hard: no efficient method known for finding the optimal solution in every instance aside from complete enumeration.

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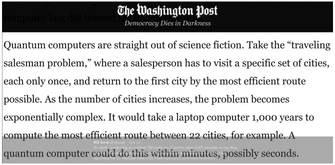
...but that doesn't mean that finding the solution to any particular instance is hard.

TSP in the Media


The Washington Post
Democracy Dies in Darkness

Quantum computers are straight out of science fiction. Take the "traveling salesman problem," where a salesperson has to visit a specific set of cities, each only once, and return to the first city by the most efficient route possible. As the number of cities increases, the problem becomes exponentially complex. It would take a laptop computer 1,000 years to compute the most efficient route between 22 cities, for example. A quantum computer could do this within minutes, possibly seconds.

From `www.twitter.com/wjcook`

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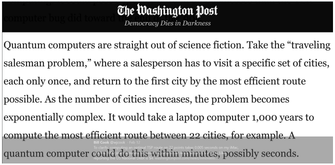
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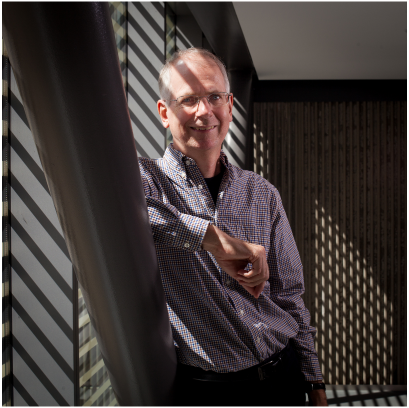
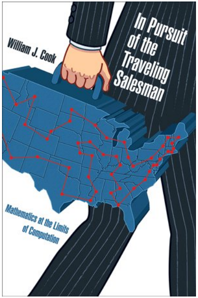
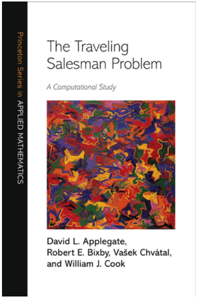
"Like reporting the US National Debt is $4" – Bill Cook

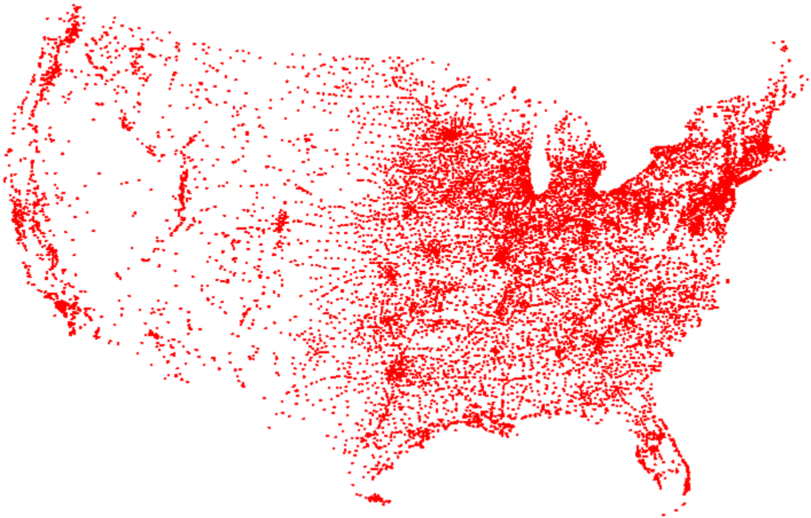
Bill Cook




William J. Cook
In Pursuit of the Traveling Salesman
Mathematics at the Limits of Computation


Princeton Series in APPLIED MATHEMATICS
The Traveling Salesman Problem
A Computational Study
David L. Applegate, Robert E. Bixby, Vašek Chvátal, and William J. Cook

The TSP: by Picture



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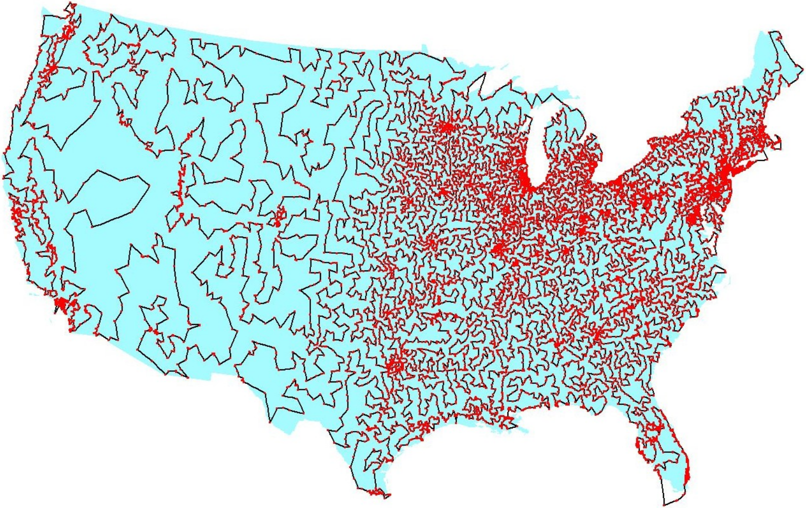

Bixby, Chvatal, Applegate, and Cook (1998)

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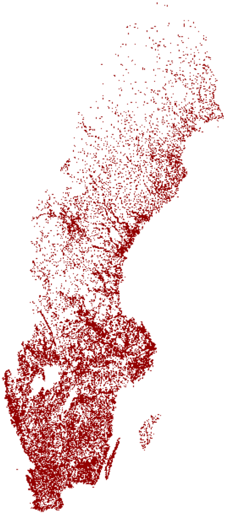

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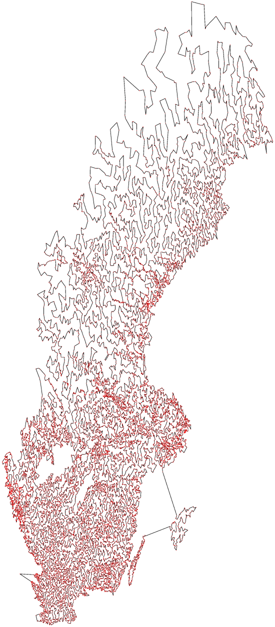

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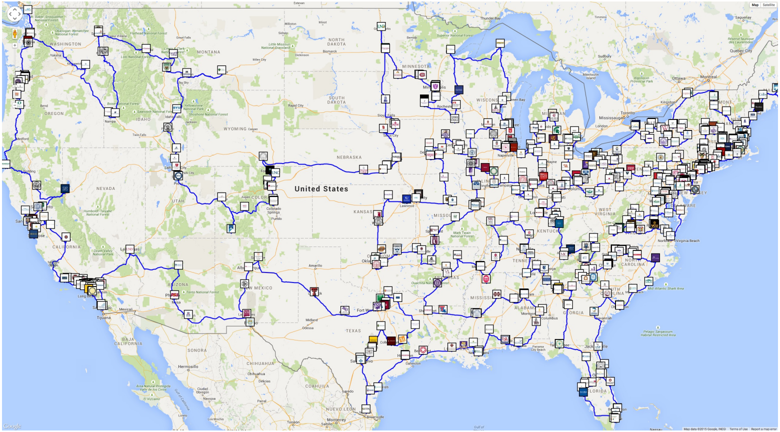

Tour of 647 college campuses from Forbes' list of America's Top Colleges

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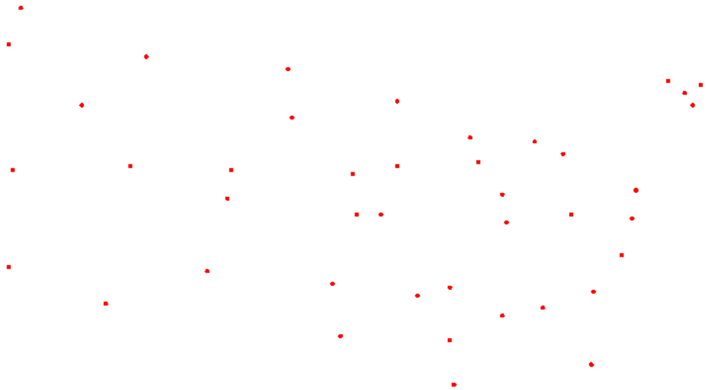

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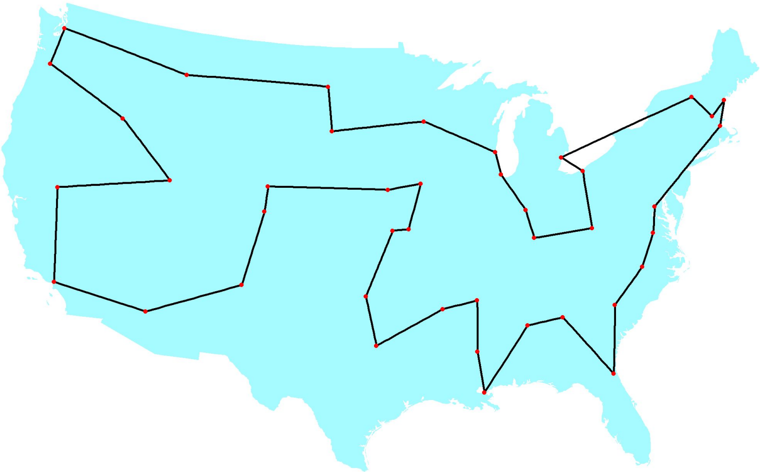

Solved by Dantzig, Fulkerson, and Johnson (1954)

Dantzig, Fulkerson, Johnson Method

- Write a linear program (LP) using variables x_e
- Idea: if $x_e = 1$ then edge e is in tour, else if $x_e = 0$ edge e is not in tour.
- Since a linear program, can only restrict $0 \leq x_e \leq 1$
- Start with linear constraints that are satisfied by any integer tour
- If solution to LP is not integer, add more constraints (*cutting planes*) satisfied by any integer tour, but not by the current LP solution.

The Subtour Elimination LP Relaxation (1954)

Let $\delta(S) := \{e = \{i,j\} : |\{i,j\} \cap S| = 1\}$ be the set of edges with exactly one endpoint in S, and let $\delta(v) := \delta(\{v\})$.

$$
\begin{array}{lll}
\text{min} & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \sum_{e \in \delta(v)} x_e = 2, & v = 1, \ldots, n \\
& \sum_{e \in \delta(S)} x_e \geq 2, & S \subset V : S \neq \emptyset, S \neq V \\
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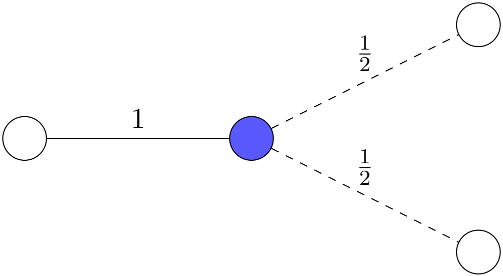

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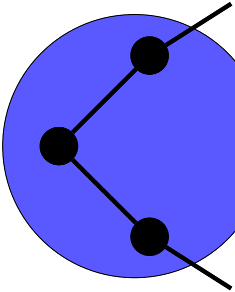

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Remarks

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- With $0 \leq x_e \leq 1$, it is a **relaxation** of the TSP and can only find cheaper solutions.

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Remarks

The closer the value of the linear program to the value of the optimal integral solution, the easier it is to find using cutting planes or other standard techniques of integer programming (such as *branch-and-bound*).

The Subtour Elimination LP Relaxation (1954)

Random Uniform Euclidean				TSPLIB			
Name	%Gap	Opttime	HKtime	Name	%Gap	Opttime	HKtime
E1k.0	0.77	1406	2.13	dsj1000	0.61	410	3.68
E1k.1	0.64	3855	2.15	pr1002	0.89	34	2.40
E1k.2	0.72	1211	2.02	si1032	0.08	25	11.32
E1k.3	0.62	956	1.92	u1060	0.65	571	3.62
E1k.4	0.69	330	1.69	vm1084	1.33	605	2.40
E1k.5	0.59	233	2.42	pcb1173	0.96	468	1.70
E1k.6	0.79	2940	1.67	d1291	1.18	27394	4.54
E1k.7	0.94	8003	1.95	rl1304	1.55	189	4.08
E1k.8	1.01	4347	1.65	rl1323	1.65	3742	4.49
E1k.9	0.61	189	2.14	nrw1379	0.43	578	2.40
E3k.0	0.71	533368	9.57	fl1400	1.74	1549	9.83
E3k.1	0.67	425631	10.54	u1432	0.29	224	2.42
E3k.2	0.74	342370	9.41	fl1577	1.66	6705	38.19
E3k.3	0.67	147135	10.30	d1655	0.94	263	6.51
E3k.4	0.73		8.07	vm1748	1.35	2224	4.43
Random Clustered Euclidean				u1817	0.90	449231	5.01
C1k.0	0.54	337	9.83	rl1889	1.55	10023	11.45
C1k.1	0.41	534	10.84	d2103	1.44	–	8.19
C1k.2	0.42	320	8.79	u2152	0.62	45205	8.10
C1k.3	0.53	214	7.63	u2319	0.02	7068	3.16
C1k.4	0.58	768	9.36	pr2392	1.22	117	5.75
C1k.5	0.58	139	9.29	pcb3038	0.81	80829	7.26
C1k.6	0.73	1247	7.07	fl3795	1.04	69886	123.66
C1k.7	0.58	449	13.24	fnl4461	0.55	–	12.47
C1k.8	0.34	140	10.40	rl5915	1.56	–	42.00
C1k.9	0.66	703	9.61	rl5934	1.38		56.15
C3k.0	0.62	16009	53.03	pla7397	0.58	–	55.42
C3k.1	0.61	17754	126.49	rl11849	1.02		102.41
C3k.2	0.70	18237	80.39	usa13509	0.66	–	120.20
C3k.3	0.57	6349	71.57	d15112	0.52		90.13
C3k.4	0.57	4845	44.02				
Random Matrices							
M1k.0	0.01	60	5.47	M3k.0	0.00	612	40.35
M1k.1	0.03	137	5.51	M3k.1	0.01	546	39.52
M1k.2	0.01	151	5.63	M10k.0	0.00	1377	367.84
M1k.3	0.01	169	5.26				

From Johnson, McGeoch 2002

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The Subtour LP bound is good in practice; what can we say about it in the worst-case?

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Integrality Gap

The *integrality gap* of an LP relaxation is the worst-case ratio (for any set of metric and symmetric edge costs) of

$$\frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}}.$$

The Subtour Elimination LP Relaxation (1954)

Theorem (Wolsey 1980, Cunningham '86, Shmoys & W '90)

The Christofides-Serdyukov algorithm produces a Hamiltonian cycle whose cost is within a factor of $\frac{3}{2}$ of the subtour LP:

$$\begin{aligned}\text{Optimal TSP Solution} &\le \text{Christofides' Cycle}\\ &\le \frac{3}{2}\text{Optimal LP Solution}\\ &\le \frac{3}{2}\text{Optimal TSP Solution.}\end{aligned}$$

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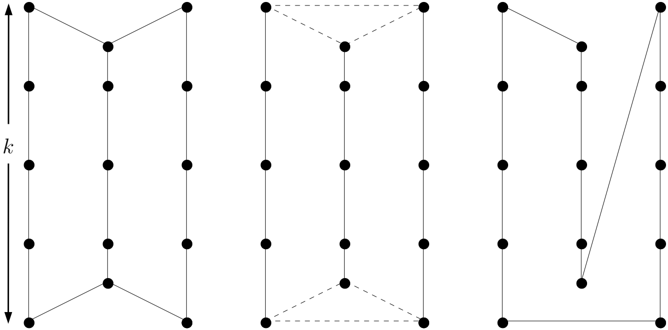
Corollary

The integrality gap of this relaxation is at most $\frac{3}{2}$. That is, for any set of metric and symmetric edge costs,

$$\frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}} \leq \frac{3}{2}.$$

The Subtour Elimination LP Relaxation (1954)



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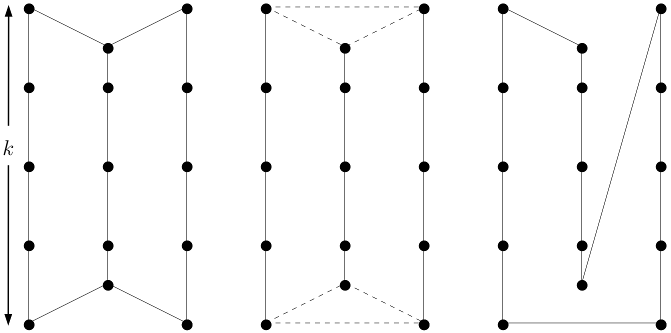
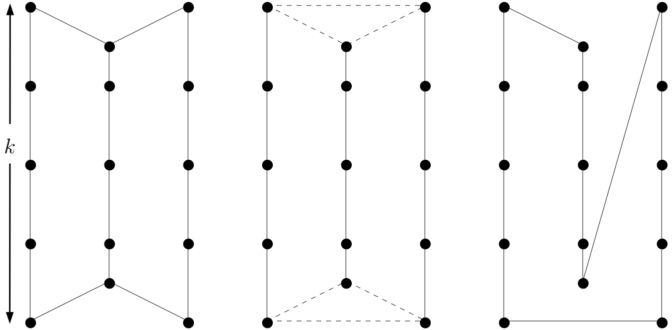


The example shows the integrality gap of this relaxation is at least 4/3. Thus, for any set of metric and symmetric edge costs,

$$\frac{4}{3} \leq \frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}} \leq \frac{3}{2}.$$

The Subtour Elimination LP Relaxation (1954)



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Open problem: Prove tight bound on integrality gap.

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Open problem: Prove tight bound on integrality gap.

Karlin, Klein, and Oveis Gharan (2020) give an algorithm that finds a tour of cost at most $\frac{3}{2} - 10^{-36}$ times the optimal cost, though they do not improve the analysis of the integrality gap.

Looking Under Rocks

Idea

Instead of LP relaxations, try SDP relaxations.

Outline

1. Introduction: The Traveling Salesman Problem and Linear Programming
2. Semidefinite Relaxations of the Traveling Salesman Problem
3. Proof Sketch: An SDP with Unbounded Integrality Gap
4. One More SDP Relaxation
5. Conclusion and Open Questions

Semidefinite Programs (SDPs)

A semidefinite program is similar to a linear program, except that we can take a matrix of variables and enforce that the matrix is positive semidefinite. Let $X \succeq 0$ denote that X is positive semidefinite.

Recall that for real symmetric X, $X \succeq 0$ if and only if

- $y^T X y \geq 0$ for all n-vectors y;
- X has all nonnegative eigenvalues.

$$
\begin{array}{lll}
\text{min} & \sum_{i,j=1}^{n} C_{ij} X_{ij} & \\
\text{subject to} & \sum_{i,j} a_{ijk} X_{ij} = b_k & k = 1, \ldots, m \\
& X \succeq 0 & \\
& X = (X_{ij}) & \text{real, symmetric}
\end{array}
$$

We can solve SDPs efficiently.

A First SDP Relaxation (1999)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs.
Let J denote the all-ones matrix, and e denote the all-ones vector.

$$
\begin{array}{lll}
\min & \frac{1}{2}\mathrm{trace}\,(CX) = \frac{1}{2}\sum_{i,j=1}^n C_{ij}X_{ij} & \\
\text{subject to} & Xe = 2e & \\
 & X_{ii} = 0, & i = 1, ..., n \\
 & 0 \le X_{ij} \le 1, & i,j = 1, ..., n \\
 & 2I - X + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0 & \\
 & X \text{ a real, symmetric } n \times n \text{ matrix.} &
\end{array}
$$

Theorem (Cvetković, Čangalović, and Kovačević-Vujčić 1999)

This semidefinite program is a relaxation of the TSP: the adjacency matrix of any Hamiltonian cycle is feasible and has the appropriate objective value.

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& X \text{ a real, symmetric } n \times n \text{ matrix.} &
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$$

X is a fractional adjacency matrix of K_n :

for $e = \{i, j\}, X_{ij} = X_{ji}$ is the proportion of edge e used.

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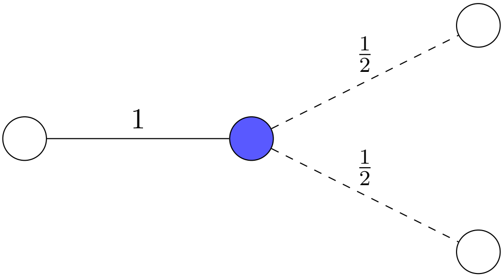

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& 2I - X + J - \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right) I \succeq 0 & \\
& X \text{ a real, symmetric } n \times n \text{ matrix.} &
\end{array}
$$

The weighted graph corresponding to X (as a weighted adjacency matrix) is at least as connected as a cycle graph, with respect to algebraic connectivity

A First SDP Relaxation (1999)

Let $C = (c_{ij})_{i,j=1}^{n}$ be the matrix of edge costs.

$$
\begin{array}{lll}
\text{min} & \frac{1}{2}\text{trace}\,(CX) & \\
\text{subject to} & Xe = 2e & \\
& X_{ii} = 0, & i = 1, \ldots, n \\
& 0 \le X_{ij} \le 1, & i, j = 1, \ldots, n \\
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& X \text{ a real, symmetric } n \times n \text{ matrix }. &
\end{array}
$$

Theorem (Goemans and Rendl, 2000)

This SDP is weaker than the Subtour Elimination LP: any feasible solution for the Subtour LP is also feasible for this SDP.

A First SDP Relaxation (1999)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs.

$$\begin{array}{lll}
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\end{array}$$

Theorem (Gutekunst and W, 2018)

This SDP has an unbounded integrality gap.

A Second SDP Relaxation (2008)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs and S^n be the set of real, symmetric $n \times n$ matrices. Also let $d = \lfloor \frac{n}{2} \rfloor$.

$$
\begin{array}{lll}
\min & \frac{1}{2}\text{trace}\left(CX^{(1)}\right) & \\
\text{subject to} & X^{(k)} \geq 0, & k = 1, \ldots, d \\
& \sum_{j=1}^{d} X^{(j)} = J - I, & \\
& I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\
& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}
$$

Theorem (de Klerk, Pasechnik, and Sotirov 2008)

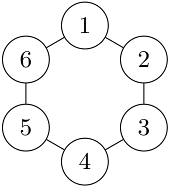
This semidefinite program is a relaxation of the TSP. Moreover, it is incomparable with the Subtour Elimination LP.

A Second SDP Relaxation (2008)

Idea

Let $\mathcal{C}$ be a Hamiltonian cycle. For $i = 1, \ldots, d = \lfloor \frac{n}{2} \rfloor$, let $X^{(i)}$ be the ith distance matrix of $\mathcal{C}$:

$$X_{jk}^{(i)} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$



$$X^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A Second SDP Relaxation (2008)

Idea

Let $\mathcal{C}$ be a Hamiltonian cycle. For $i = 1, ..., d = \lfloor \frac{n}{2} \rfloor$, let $X^{(i)}$ be the ith distance matrix of $\mathcal{C}$:

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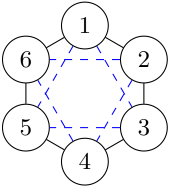

$$X^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

A Second SDP Relaxation (2008)

$$
\begin{array}{lll}
\min & \frac{1}{2}\mathrm{trace}\left(CX^{(1)}\right) & \\
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& I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\
& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}
$$

For $i = 1, \ldots, d = \lfloor \frac{n}{2} \rfloor$, these quickly follow from

$$
X^{(i)}_{jk} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}
$$

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\end{array}
$$

- The distance matrices of a cycle form an *association scheme.*
- This is an application of a more general statement about association schemes.

(See de Klerk, Filho, Pasechnik 2012)

- The distance matrices of a cycle are *circulant matrices.*
- Linear combinations of circulant matrices are circulant.
- Circulant matrices have well-understood eigenvalues.

(see Gutekunst and W. 2018)

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& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}$$

$$\begin{pmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} \\
m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\
m_{n-2} & m_{n-1} & m_0 & \ddots & m_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_1 & m_2 & m_3 & \cdots & m_0
\end{pmatrix}$$

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A Second SDP Relaxation (2008)

Goal

For $X^{(j)}_{st} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \qquad k = 1, \ldots, d.$$

$$\begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & \ddots & m_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_0 \end{pmatrix}$$

For $\omega_n = e^{-\frac{2\pi i}{n}}$,

$$\lambda_t(M) = \sum_{s=0}^{n-1} m_s \omega_n^{st}, \quad t = 1, ..., n-1, \quad \lambda_n(M) = \sum_{s=0}^{n-1} m_s.$$

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$$\begin{pmatrix} 1 & \cos(2\pi k/n) & \cos(2\pi 2k/n) & \cdots & \cos(2\pi 2k/n) & \cos(2\pi k/n) \\ \cos(2\pi k/n) & 1 & \cos(2\pi k/n) & \cdots & \cos(2\pi 3k/n) & \cos(2\pi 2k/n) \\ \cos(2\pi 2k/n) & \cos(2\pi k/n) & 1 & \ddots & \cos(2\pi 4k/n) & \cos(2\pi 3k/n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos(2\pi k/n) & \cos(2\pi 2k/n) & \cos(2\pi 3k/n) & \cdots & \cos(2\pi k/n) & 1 \end{pmatrix}$$

$$\lambda_t(M) = \sum_{s=0}^{n-1} m_s \omega_n^{st}, \quad t = 1, ..., n-1, \quad \lambda_n(M) = \sum_{s=0}^{n-1} m_s.$$

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For $X^{(j)}_{st} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \qquad k = 1, \ldots, d.$$

For $t \leq n$,

$$\begin{aligned}
\lambda_t(M) &= \sum_{s=0}^{n-1} m_s \omega_n^{st} \\
&= 1 + \cos\left(\frac{2\pi kd}{n}\right) \omega_n^{dt} + \sum_{s=1}^{d-1} \cos\left(\frac{2\pi sk}{n}\right) \left(\omega_n^{st} + \omega_n^{(n-s)t}\right) \\
&= \cdots \\
&= \begin{cases} 2d, & \text{if } k = t = d \\ d, & \text{if } k \neq d, t \in \{k, n-k\} \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

A Second SDP Relaxation (2008)

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$$
\begin{array}{lll}
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\end{array}
$$

Theorem (Gutekunst and W, 2018)

This SDP has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that

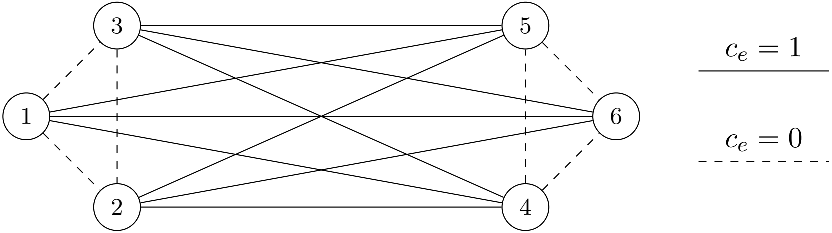
$$\frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{SDP}}(C)} \leq \alpha$$

for all cost matrices C with metric, symmetric edge costs.

Our Main Theorem: Proof Sketch

Let n be even and consider the cost matrix

$$\hat{C} := \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d.$$



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$\hat{C}$ corresponds to:

- a *cut semimetric*: costs where, for some $S \subset V$, $c_{ij} = 1$ if $\{i, j\} \in \delta(S)$ and $c_{ij} = 0$ otherwise.
- an instance of *Euclidean TSP*: vertices $1, \ldots, \frac{n}{2}$ are at $0 \in \mathbb{R}^1$ and vertices $\frac{n}{2} + 1, \ldots, n$ are at $1 \in \mathbb{R}^1$. Costs are given by the Euclidean distance between corresponding vertices.

Our Main Theorem: Proof Sketch

Theorem (Gutekunst and W, 2018)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Corollary

There exists no constant $\alpha > 0$ such that

$$\frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{SDP}}(C)} \leq \alpha$$

for all cost matrices C with metric, symmetric edge costs.

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Strategy:

1. Look within a class of feasible solutions that respect the symmetry of $\hat{C}$.
2. Exploit the structure of such solutions by reducing the SDP to an LP *for solutions in that class.*
3. Find a feasible solution to the LP achieving the desired cost.

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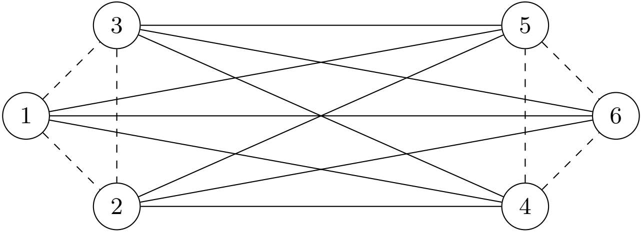
For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Candidate solutions:

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d\right) - a_j I_n, \qquad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$



$X_e^{(j)} = b_j$ (solid line)

$X_e^{(j)} = a_j$ (dashed line)

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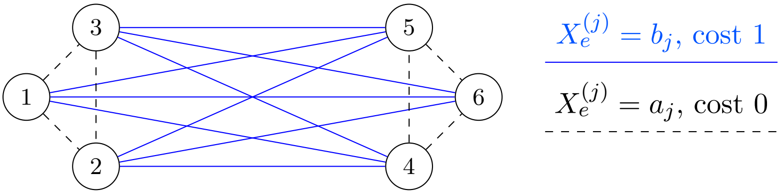

TSP Solutions

$$\mathrm{OPT}_{\mathrm{TSP}}(\hat{C}) = 2$$

SDP Solutions

$$\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) = \frac{1}{2}\mathrm{trace}\left(CX^{(1)}\right)$$

$$= 0 \times 2\binom{n/2}{2}a_1 + 1 \times \left(\frac{n}{2}\right)^2 b_1$$

Our Main Theorem: Proof Sketch

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For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Let

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \qquad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$

Want to verify that it satisfies

$$\begin{array}{ll} X^{(k)} \geq 0, & k = 1, \ldots, d \\ \sum_{j=1}^{d} X^{(j)} = J - I, & \\ I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\ X^{(k)} \in S^n, & k = 1, \ldots, d, \end{array}$$

so need $a_j \geq 0$, $b_j \geq 0$, $\sum_{j=1}^{d} a_j = 1$.

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The SDP constraint $I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0$ becomes

$$\left(\begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d\right) + (1 - a^{(k)}) I_n \succeq 0,$$

where $a^{(k)}$ and $b^{(k)}$ are linear combinations of $a_1, \ldots, a_d$.

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$$a^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i, \qquad b^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) b_i.$$

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- The eigenvalues of $A \otimes B$ are $\lambda_i(A)\lambda_j(B)$.
- J_d has one eigenvalue d, all other eigenvalues are zero.
- The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are $a + b$ and $a - b$.

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So eigenvalues are

$$1 - a^{(k)}, \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} + b^{(k)}\right), \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} - b^{(k)}\right).$$

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for

$$a^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i, \qquad b^{(k)} = -\left(1 - \frac{2}{n}\right) a^{(k)} - \frac{2}{n}.$$

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Intermediate step: Rewriting $b^{(k)}$ in terms of $a^{(k)}$, and imposing that the eigenvalues, a_j, and b_j, are nonnegative, and finding minimum-cost solution becomes linear program:

$$
\begin{array}{llll}
\max & a_1 & & \\
\text{subject to} & \sum_{i=1}^{d} \cos\left(\frac{2\pi i k}{n}\right) a_i & \geq -\frac{2}{n-2}, & k = 1, \ldots, d \\
& \sum_{i=1}^{d} \cos\left(\frac{2\pi i k}{n}\right) a_i & \leq 1, & k = 1, \ldots, d \\
& \sum_{i=1}^{d} a_i & = 1 & \\
& a_i & \leq \frac{4}{n-2}, & i = 1, \ldots, d-1 \\
& a_d & \leq \frac{2}{n-2} & \\
& a_i & \geq 0, & i = 1, \ldots, d.
\end{array}
$$

Our Main Theorem: Proof Sketch

Theorem (Gutekunst and W, 2018)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

$$
\begin{array}{lllr}
\max & a_1 & & \\
\text{subject to} & \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i & \geq -\frac{2}{n-2}, & k = 1, \ldots, d \\
& \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i & \leq 1, & k = 1, \ldots, d \\
& \sum_{i=1}^{d} a_i & = 1 & \\
& a_i & \leq \frac{4}{n-2}, & i = 1, \ldots, d-1 \\
& a_d & \leq \frac{2}{n-2} & \\
& a_i & \geq 0, & i = 1, \ldots, d.
\end{array}
$$

Guess and verify that the following solution is feasible.

$$
a_j = \frac{2}{n-2}\left(\cos\left(\frac{\pi j}{d}\right) + 1\right), \qquad j = 1, \ldots, d.
$$

Our Main Theorem: Proof Sketch

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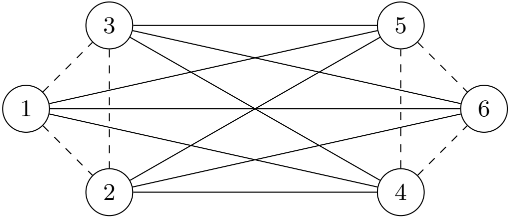
Thus we find solutions where



— $b_1 = \frac{1-\cos\left(\frac{\pi}{d}\right)}{n} \sim \frac{1}{n^3}$

- - - $a_1 = \frac{2\cos\left(\frac{\pi}{d}\right)+2}{n-2}$

$$\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{n^2}{4} b_1 \sim \frac{1}{n}.$$

Summary

- The 2008 SDP relaxation has an unbounded integrality gap
- To show that it produces arbitrarily bad solutions, we:
  1. Looked within a class of feasible solutions that respect the symmetry of $\hat{C}$.
  2. Exploited the structure of such solutions by reducing the SDP to an LP *for solutions in that class*.
  3. Found a feasible solution to the LP achieving the whose cost decreases like $\frac{1}{n^3}$.

Corollaries of Our Theorem

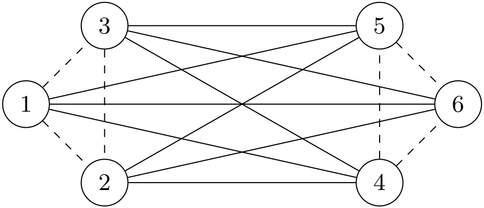
Theorem (Gutekunst and W, 2018)

The SDP has an unbounded integrality gap.

Corollary

The SDP is non-monotonic, unlike the TSP and subtour elimination LP.

We've found SDP solutions costing $\frac{n^2}{4} b_1 \approx \frac{1}{n}$, which become arbitrarily small with n



— $b_1 = \frac{1-\cos\left(\frac{\pi}{d}\right)}{n} \sim \frac{1}{n^3}$

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Corollaries of Our Theorem

Theorem (Gutekunst and W, 2018)

The SDP has an unbounded integrality gap.

Corollary

The earlier SDP of Cvetković, Čangalović, and Kovačević-Vujčić has an unbounded integrality gap: the same $X^{(1)}$ we found is feasible (and has exactly the same algebraic connectivity as a cycle).

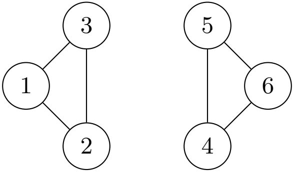
Corollaries of Our Theorem

Theorem (Gutekunst and W, 2018)

The SDP has an unbounded integrality gap.

Corollary

A related SDP from de Klerk, de Oliveira Filho, and Pasechnik 2012 for the k-cycle cover problem also has an unbounded integrality gap.



A Third SDP Relaxation (2012)

De Klerk and Sotirov (2012) introduce one more SDP relaxation based on an SDP relaxation of the *quadratic assignment problem* (QAP) due to Povh and Rendl (2009).

Idea of the QAP version: let $X \in \Pi_n$ be $n \times n$ permutation matrix, with $X_{ij} = 1$ iff the ith city we visit is j, for some ordering of the tour. Then

$$X^T A^{(n)} X$$

gives the adjacency matrix of a tour, where $A^{(n)}$ is the adjacency matrix of the tour $1, 2, 3, \ldots, n$, and its cost is

$$\frac{1}{2}\mathrm{trace}\left(A^{(n)} X C X^T\right) = \frac{1}{2}\left\langle X^T A^{(n)} X, C \right\rangle .$$

A Third SDP Relaxation (2012)

Idea: Create a matrix

$$Y = \begin{pmatrix} Y^{(11)} & Y^{(12)} & \dots & Y^{(1n)} \\ Y^{(21)} & Y^{(22)} & \dots & Y^{(2n)} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{(n1)} & Y^{(n2)} & \dots & Y^{(nn)} \end{pmatrix},$$

where $Y^{(ij)} = X_i X_j^T$, for X_i the ith column of X, and $Y^{(ij)} = E_{st}$ for some s, t, where E_{st} the matrix of all 0s, with one 1 in the s, t entry.

Also, $Y^{(ii)} = E_{kk}$ for some k, and $Y^{(ii)} \neq Y^{(jj)}$ for $i \neq j$.

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Also, $Y^{(ii)} = E_{kk}$ for some k, and $Y^{(ii)} \neq Y^{(jj)}$ for $i \neq j$.

Finally, $Y = vec(X)vec(X)^T$, where $vec(X)$ converts X to a vector by stacking its columns.

A Third SDP Relaxation (2012)

The Povh and Rendl (2009) relaxation is

$$
\begin{array}{lll}
\min & \frac{1}{2}\mathrm{trace}\left(\left(C \otimes A^{(n)}\right) Y\right) & \\
\text{subject to} & \mathrm{trace}((I_n \otimes E_{jj}^{(n)})Y) = 1 & j = 1, \ldots, n \\
& \mathrm{trace}((E_{jj}^{(n)} \otimes I_n)Y) = 1 & j = 1, \ldots, n \\
& \mathrm{trace}((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0 & \\
& \mathrm{trace}(J_{n^2} Y) = n^2 & \\
& Y \geq 0, Y \succeq 0, Y \in \mathbb{S}^{n^2 \times n^2}. &
\end{array}
$$

A Third SDP Relaxation (2012)

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$$\begin{array}{lll} \min & \frac{1}{2}\mathrm{trace}\left(\left(C \otimes A^{(n)}\right) Y\right) & \\ \text{subject to} & \mathrm{trace}((I_n \otimes E_{jj}^{(n)})Y) = 1 & j = 1, \ldots, n \\ & \mathrm{trace}((E_{jj}^{(n)} \otimes I_n)Y) = 1 & j = 1, \ldots, n \\ & \mathrm{trace}((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0 & \\ & \mathrm{trace}(J_{n^2} Y) = n^2 & \\ & Y \geq 0, Y \succeq 0, Y \in \mathbb{S}^{n^2 \times n^2}. & \end{array}$$

Theorem (de Klerk, Pasechbik, Sotirov 2008; Povh & Rendl, 2009)

This SDP has the same optimal value as the SDP of de Klerk, Pasechnik, and Sotirov.

A Third SDP Relaxation (2012)

De Klerk and Sotirov (2012) apply *symmetry reduction*: assume $X_{11} = 1$ in the permutation matrix and derive the associated SDP relaxation as before.

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Computational results are again promising: better than the subtour LP on small instances of the TSP.

A Third SDP Relaxation (2012)

Theorem (Gutekunst & W)

Previous instances give an integrality gap of at least 2 for the de Klerk-Sotirov SDP relaxation.

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Theorem (Gutekunst & W)

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Theorem (Gutekunst & W)

For any constant c, can prove an integrality gap of at least c for the de Klerk-Sotirov SDP relaxation.

Idea: We generalize our previous instances to a *simplicial instances* on g groups of n/g vertices: cost 0 for edges within each group, cost 1 for edges between groups.

Open Questions

1. How does this SDP perform on special cases of the TSP?

- We've shown that the integrality gap is unbounded on the general metric and symmetric TSP, as well as on Euclidean TSP.
- On *graphic* TSP (where edge costs correspond to shortest paths in a connected input graph), the integrality gap is at most 2. Is it strictly better?

Open Questions

1. How does this SDP perform on special cases of the TSP?
2. If you combine both this SDP and the subtour LP, can you guarantee an integrality gap of $1.5 - \epsilon$ for any $\epsilon > 0$?

Big Open Questions

Open Problem

Prove tight bound on integrality gap of Subtour LP.

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Open Problem

Is there some other way of understanding the surprising practical effectiveness of the Subtour LP?

Samuel C. Gutekunst and David P. Williamson, The Unbounded Integrality Gap of a Semidefinite Relaxation of the Traveling Salesman Problem, *SIAM Journal on Optimization* 28:2073–2096, 2018.

Samuel C. Gutekunst and David P. Williamson, Semidefinite Programming Relaxations of the Traveling Salesman Problem and Their Integrality Gaps, To appear, *Mathematics of Operations Research*.

Thanks for your attention.