A Simple ¾-Approximation Algorithm for MAX SAT

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Maximum Satisfiability

- Input:

  n Boolean variables $x_1, \ldots, x_n$

  m clauses $C_1, \ldots, C_m$ with weights $w_j \geq 0$

  – each clause is a disjunction of literals,

    e.g. $C_1 = x_1 \lor x_2 \lor \bar{x}_3$

- Goal: truth assignment to the variables that maximizes the weight of the satisfied clauses

Approximation Algorithms

- An α-approximation algorithm runs in polynomial time and returns a solution of at least α times the optimal.

- For a randomized algorithm, we ask that the expected value is at least α times the optimal.

A ½-approximation algorithm

- Set each x_i to true with probability ½.
- Then if l_j is the number of literals in clause j

$$
\begin{aligned}
& E[\text{Weight satisfied clauses}] \\
& \quad = \sum_{j=1}^{m} w_j \Pr[\text{Clause } j \text{ satisfied}] \\
& \quad = \sum_{j=1}^{m} w_j \left(1 - \left(\frac{1}{2}\right)^{\ell_j}\right) \\
& \quad \geq \frac{1}{2} \sum_{j=1}^{m} w_j \geq \frac{1}{2} OPT.
\end{aligned}
$$

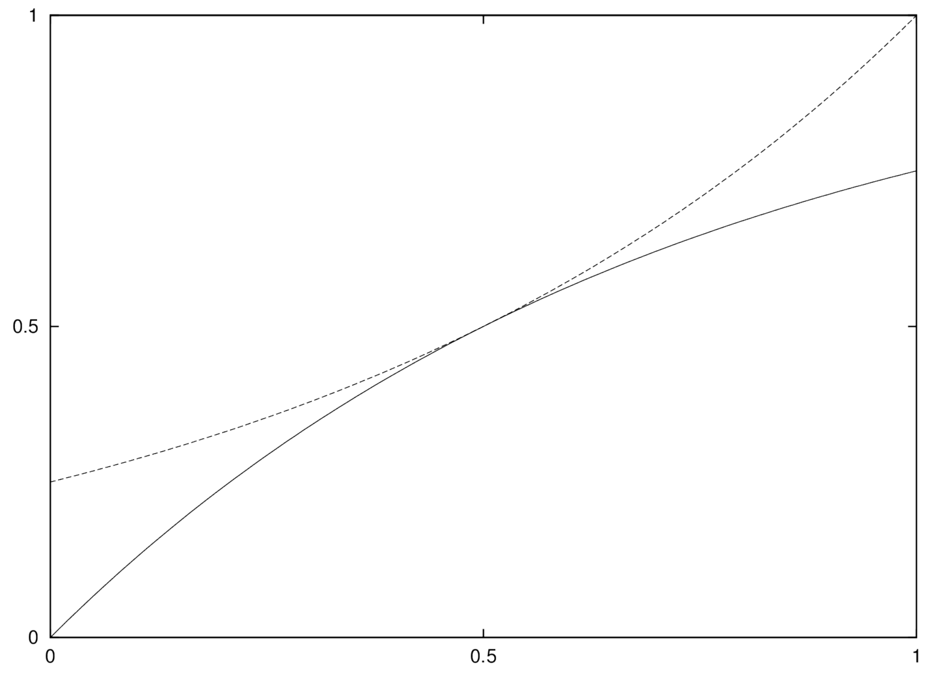
What about a deterministic algorithm?

- Use the *method of conditional expectations* (Erdős and Selfridge '73, Spencer '87)
- If $E[W|x_1 \leftarrow true] \geq E[W|x_1 \leftarrow false]$ then set x_1 true, otherwise false.
- Similarly, if X_{i-1} is event of how first $i-1$ variables are set, then if $E[W|X_{i-1}, x_i \leftarrow true] \geq E\left[W \middle| X_{i-1,} x_i \leftarrow false\right]$, set x_i true.
- Show inductively that $E[W|X_i] \geq E[W] \geq \frac{1}{2}$ OPT.

An LP relaxation

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\
\text{subject to} & \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j, & \forall C_j = \displaystyle\bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i, \\
& 0 \leq y_i \leq 1, & i = 1, \ldots, n, \\
& 0 \leq z_j \leq 1, & j = 1, \ldots, m.
\end{array}
$$

Nonlinear randomized rounding



(Goemans, W 94) Pick any function f such that $1-4^{-x} \leq f(x) \leq 4^{x-1}$. Set x_i true with probability $f(y_i^*)$, where y^* is an optimal LP solution.

Analysis

$$\begin{aligned}
\Pr[\text{clause } C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - f(y_i^*)) \prod_{i \in N_j} f(y_i^*) \\
&\leq \prod_{i \in P_j} 4^{-y_i^*} \prod_{i \in N_j} 4^{y_i^* - 1} \\
&= 4^{-\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*)\right)} \\
&\leq 4^{-z_j^*}.
\end{aligned}$$

$$\begin{aligned}
E[W] &\geq \sum_{j=1}^{m} w_j \Pr[\text{clause } C_j \text{ satisfied}] \\
&\geq \sum_{j=1}^{m} w_j \left(1 - 4^{-z_j^*}\right) \\
&\geq \frac{3}{4} \sum_{j=1}^{m} w_j z_j^* \geq \frac{3}{4} OPT.
\end{aligned}$$

Integrality gap

$$
\begin{array}{llll}
\text{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & & \\
\text{subject to} & \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j, & & \forall C_j = \displaystyle\bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i, \\
& 0 \leq y_i \leq 1, & & i = 1, \ldots, n, \\
& 0 \leq z_j \leq 1, & & j = 1, \ldots, m.
\end{array}
$$

$$x_1 \vee x_2, \quad \bar{x}_1 \vee x_2, \quad x_1 \vee \bar{x}_2, \quad \bar{x}_1 \vee \bar{x}_2$$

The result is tight since LP solution $z_1 = z_2 = z_3 = z_4 = 1$ and $y_1 = y_2 = \frac{1}{2}$ feasible for instance above, but OPT = 3.

Chan, Lee, Raghavendra, Steurer (STOC 13) show no superpolynomially sized LP can give a better integrality gap.

Current status

- NP-hard to approximate better than 0.875 (Håstad '01)

- Combinatorial approximation algorithms
  - Johnson's algorithm (1974): Simple ½-approximation algorithm (Greedy version of the randomized algorithm)
  - Improved analysis of Johnson's algorithm: $^2/_3$-approx. guarantee [Chen, Friesen, Zheng '99, Engebretsen '04]
  - Randomizing variable order improves guarantee slightly [Costello, Shapira, Tetali SODA 11]

- Algorithms using Linear or Semidefinite Programming
  - Yannakakis '94, Goemans, W '94:

Question [W '98]: **Is it possible to obtain a 3/4-approximation algorithm without solving a linear program?**

(Selected) results

- Poloczek, Schnitger (SODA 11):
  - "randomized Johnson" – combinatorial ¾-approximation algorithm
- Van Zuylen (WAOA 11):
  - Simplification of "randomized Johnson" probabilities and analysis
- Buchbinder, Feldman, Naor, and Schwartz (FOCS 12):
  - Another ¾-approximation algorithm for MAX SAT as a special case of submodular function maximization
  - Can be shown that their MAX SAT alg is equivalent to van Zuylen's.

(Selected) results

- Poloczek, Schnitger '11
- Van Zuylen '11
- Buchbinder, Feldman, Naor and Schwartz '12

Common properties:

- iterately set the variables in an "online" fashion,
- the probability of setting x_i to true depends on clauses containing x_i or $\bar{x}_i$ that are still undecided.

Today

- Give "textbook" version of Buchbinder et al.'s algorithm with an even simpler analysis (Poloczek, van Zuylen, W, LATIN 14)
- Give a simple deterministic version of the algorithm (Poloczek, Schnitger, van Zuylen, W, manuscript)
- Give an experimental analysis that shows that the algorithm works very well in practice (Poloczek, W, SEA 2016)

Buchbinder et al.'s approach

- Keep two bounds on the solution
  - **Lower bound LB** = weight of clauses already satisfied
  - **Upper bound UB** = weight of clauses not yet unsatisfied

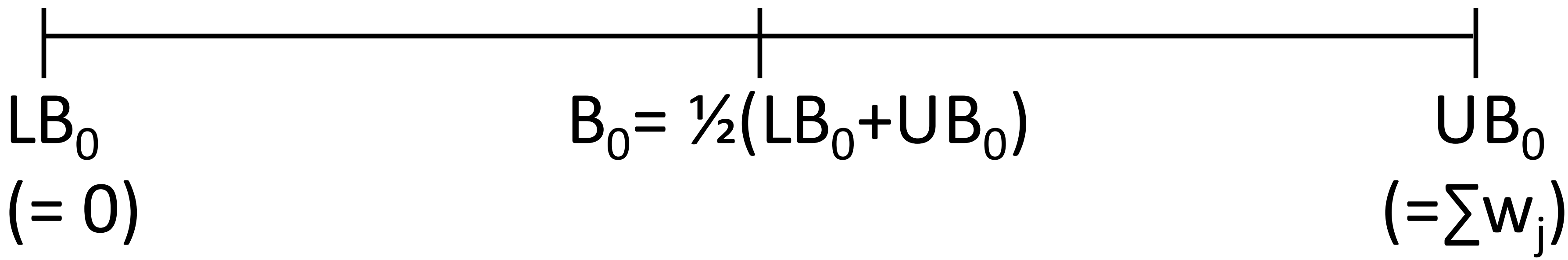
- Greedy can focus on two things:
  - maximize **LB**,
  - maximize **UB**,

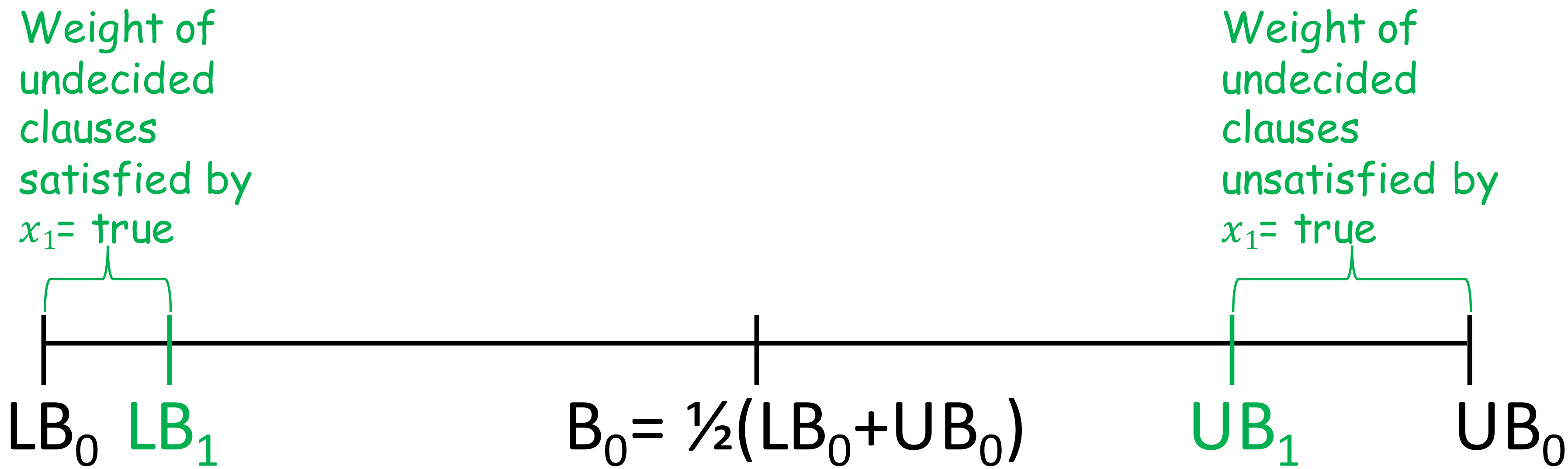
  but either choice has bad examples…

  E.g. $x_1 \vee x_2$ (wt $1+\varepsilon$), $\overline{x}_1$ (wt 1)

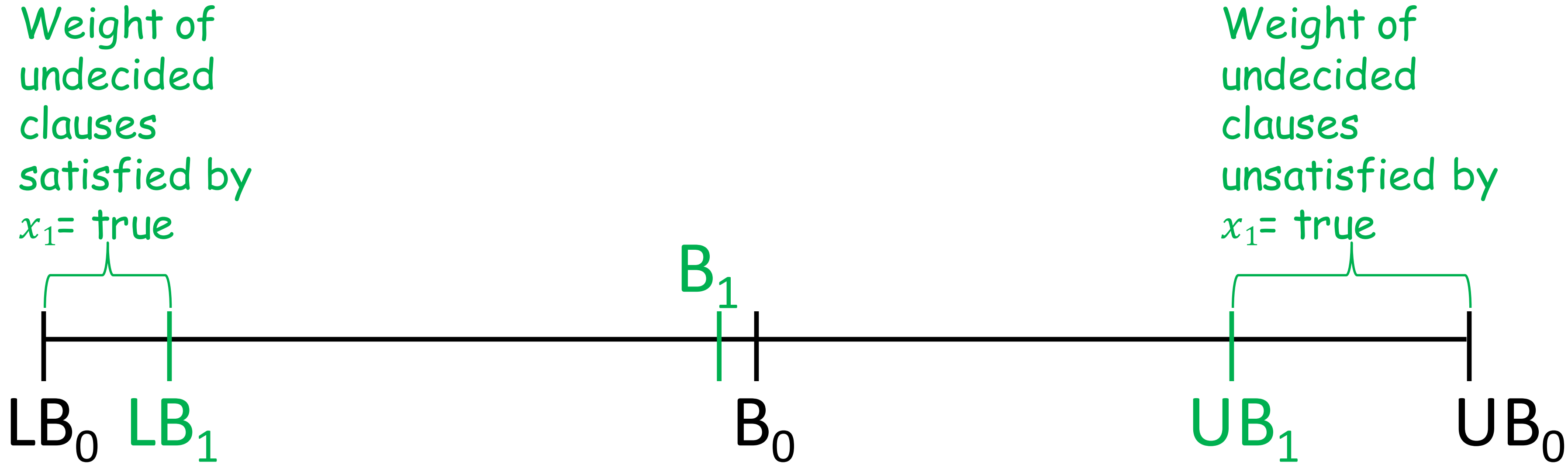
  $x_1 \vee x_2$ (wt $1+\varepsilon$), $\overline{x}_1$ (wt ε), $\overline{x}_2$ (wt 1)

- Key idea: make choices to increase **B = ½ (LB+UB)**

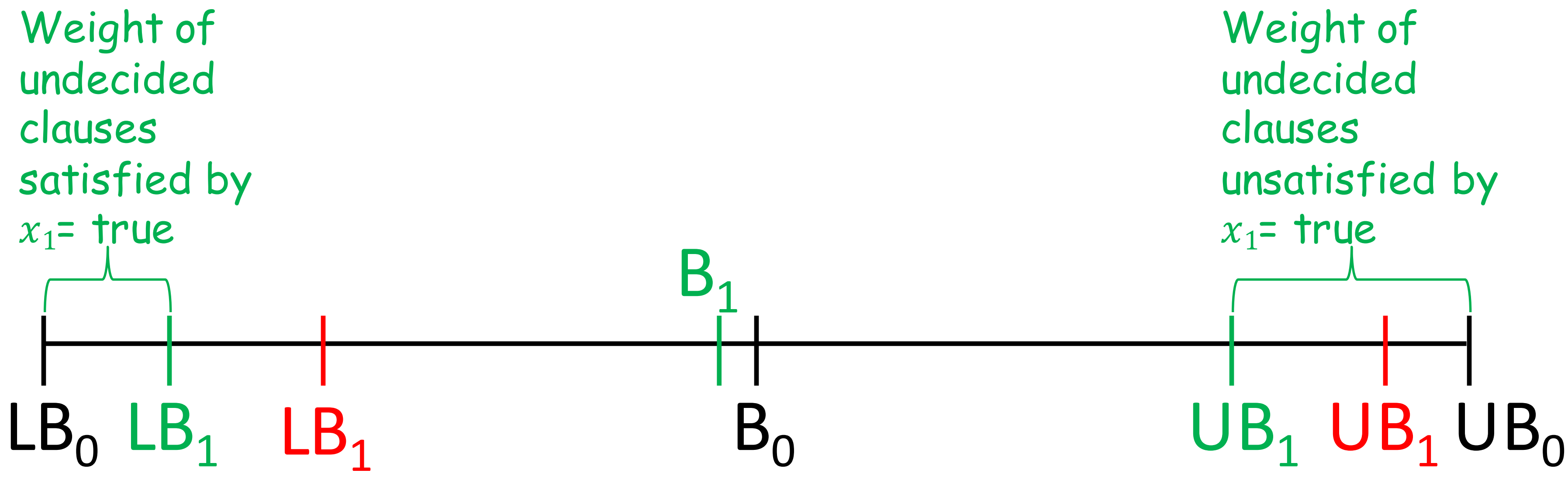

LB_0
$(= 0)$
$B_0 = ½(LB_0 + UB_0)$
UB_0
$(= \sum w_j)$



Set x_1 to true



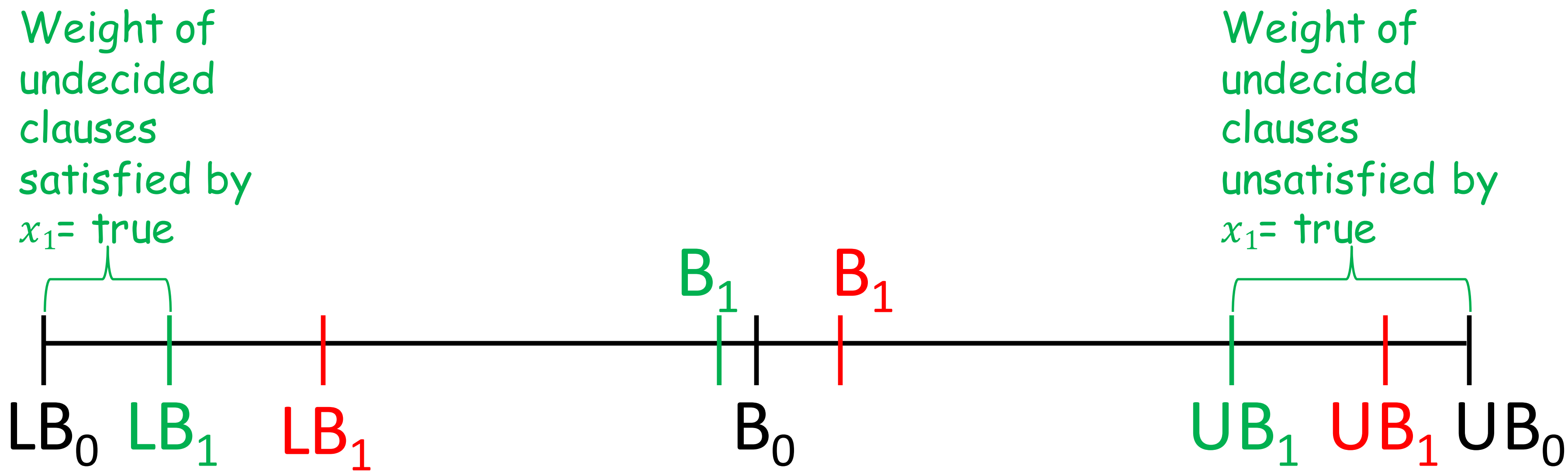
Set x_1 to true



Set x_1 to true

or

Set x_1 to false



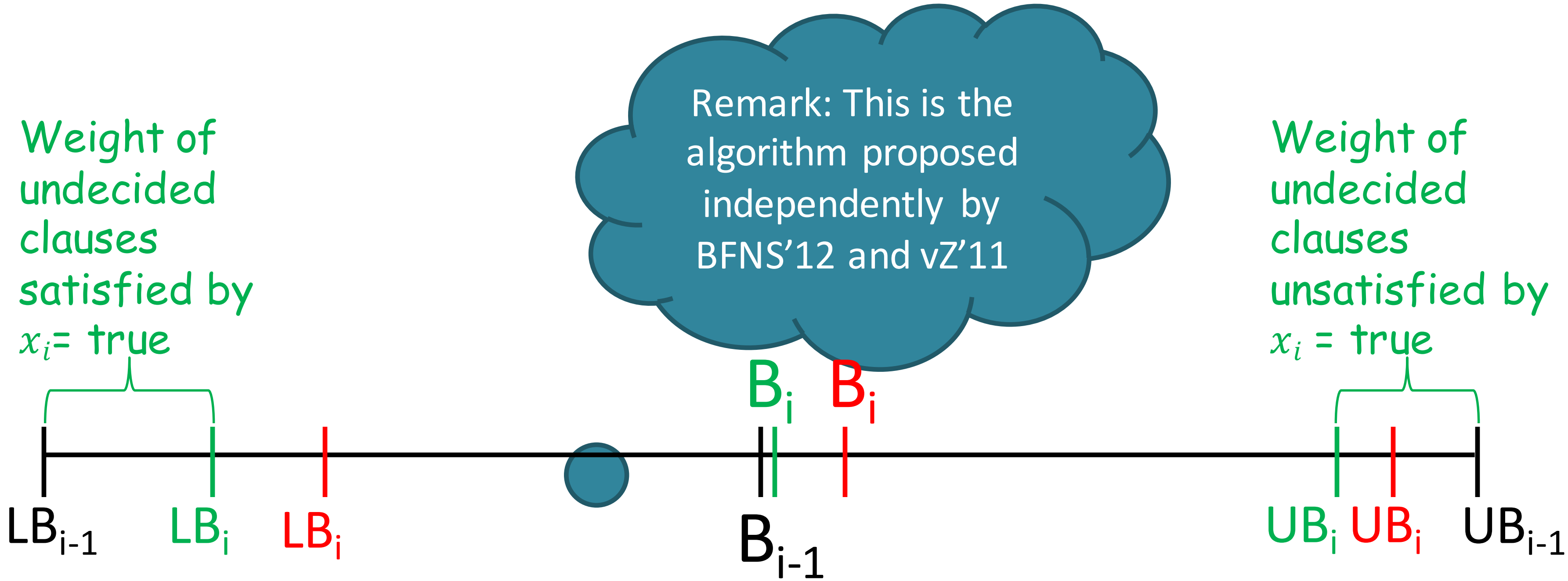
Set x_1 to true

or

Set x_1 to false

Guaranteed that

$$\underbrace{(B_1 - B_0)}_{t_1} + \underbrace{(B_1 - B_0)}_{f_1} \geq 0$$



<u>Algorithm</u>:

- if $t_i < 0$, set x_i to false
- if $f_i < 0$, set x_i to true
- else, set x_i to true with probability $\frac{t_i}{t_i+f_i}$

$$\underbrace{(B_i - B_{i-1})}_{t_i} + \underbrace{(B_i - B_{i-1})}_{f_i} \geq 0$$

Example

Initalize:

- LB = 0
- UB = 6

Step 1:

- $t_1 = \frac{1}{2}(\triangle LB + \triangle UB) = \frac{1}{2}(1 + (-2)) = -\frac{1}{2}$
- $f_1 = \frac{1}{2}(\triangle LB + \triangle UB) = \frac{1}{2}(2 + 0) = 1$
- Set x_1 to false

Clause	Weight
$\bar{x}_1$	2
$x_1 \vee x_2$	1
$\bar{x}_2 \vee x_3$	3

Example

Clause	Weight
$\bar{x}_1$	2
$\cancel{x_1 \vee}\, x_2$	1
$\bar{x}_2 \vee x_3$	3

Step 2:

- $t_2 = \frac{1}{2}(\triangle LB + \triangle UB) = \frac{1}{2}(1+0) = \frac{1}{2}$
- $f_2 = \frac{1}{2}(\triangle LB + \triangle UB) = \frac{1}{2}(3+(-1)) = 1$
- Set x_2 to true with probability 1/3 and to false with probability 2/3

Example

Clause	Weight
$\bar{x}_1$	2
$x_1 \vee x_2$	1
$\bar{x}_2 \vee x_3$	3

Algorithm's solution:

$x_1 =$ false

$x_2 =$ true w.p. 1/3 and false w.p. 2/3

$x_3 =$ true

Expected weight of satisfied clauses: $5\frac{1}{3}$

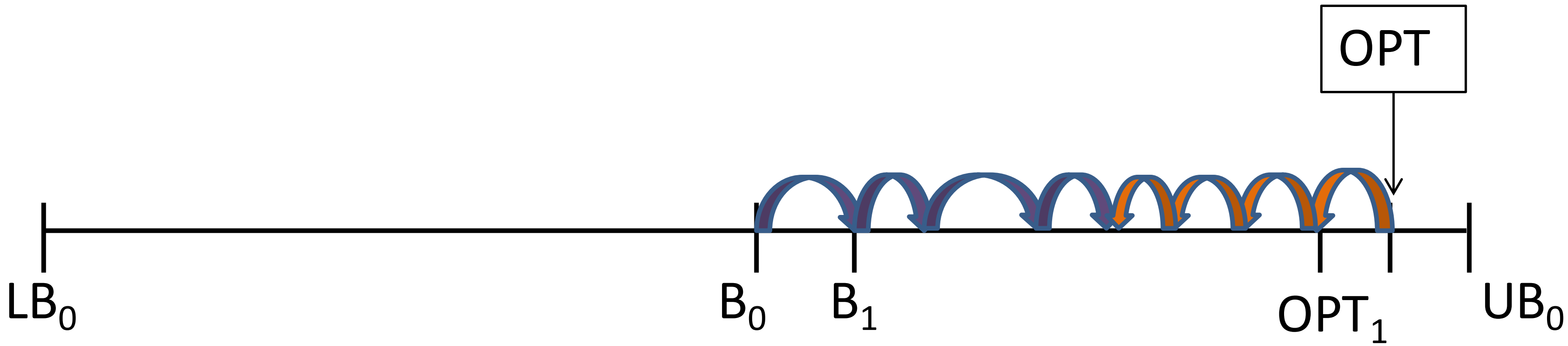
Relating Algorithm to Optimum

Let $x_1^*, x_2^*, \dots, x_n^*$ be an optimal truth assignment

Let OPT_i = weight of clauses satisfied if setting $x_1, \dots, x_i$ as the algorithm does, and $x_{i+1} = x_{i+1}^*, \dots, x_n = x_n^*$

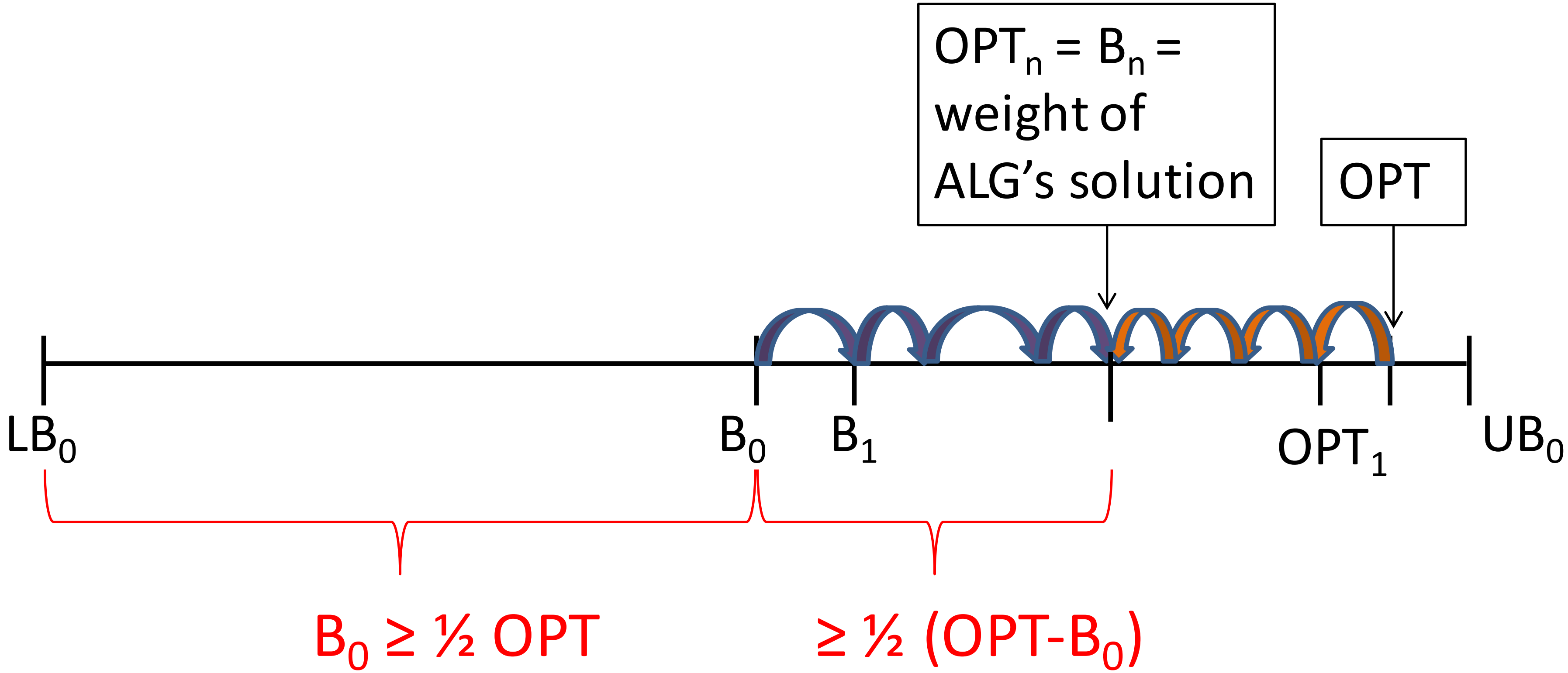
Key Lemma:

$$E[B_i - B_{i-1}] \geq E[OPT_{i-1} - OPT_i]$$



Key Lemma:

$$E[B_i - B_{i-1}] \geq E[OPT_{i-1} - OPT_i]$$

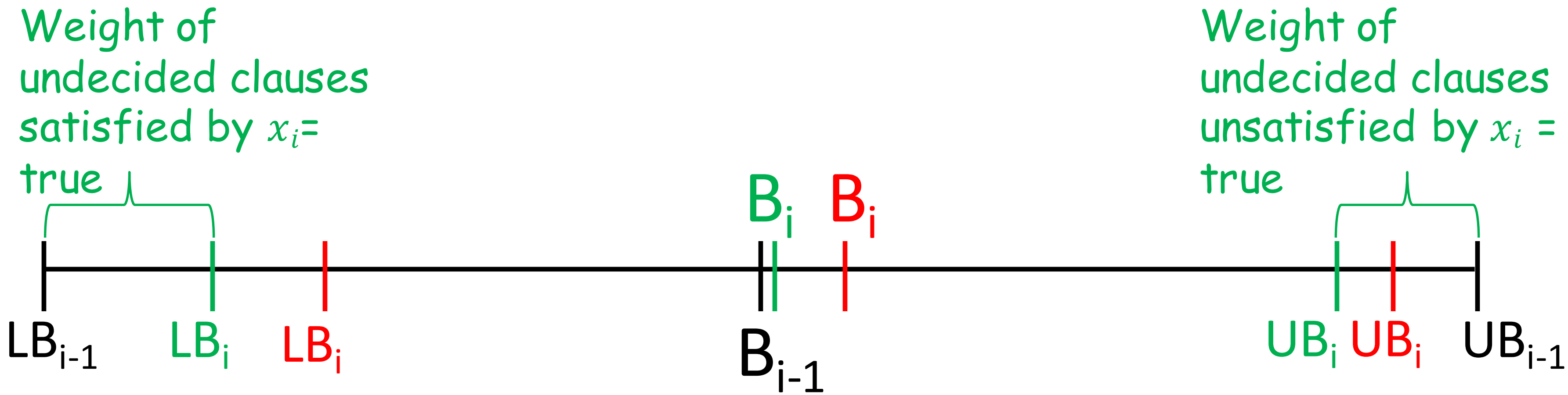


Key Lemma:

Conclusion: expected weight of ALG's solution is

$$E[B_n] \geq B_0 + \frac{1}{2}(OPT - B_0) = \frac{1}{2}(OPT + B_0) \geq \frac{3}{4} OPT$$

Relating Algorithm to Optimum



Suppose x_i^*= true

If algorithm sets x_i to true,

- $B_i - B_{i-1} = t_i$
- $OPT_{i-1} - OPT_i = 0$

If algorithm sets x_i to false,

- $B_i - B_{i-1} = f_i$
- $OPT_{i-1} - OPT_i \leq LB_i - LB_{i-1} + (UB_i - UB_{i-1})$

$$= 2(B_i - B_{i-1}) = 2t_i$$

Want to show:

Key Lemma:

$$E[B_i - B_{i-1}] \geq E[OPT_{i-1} - OPT_i]$$

Relating Algorithm to Optimum

Want to show:

<u>Key Lemma</u>:
$$E[B_i - B_{i-1}] \geq E[OPT_{i-1} - OPT_i]$$

Know:

If algorithm sets x_i to true,

- $B_i - B_{i-1} = t_i$
- $OPT_{i-1} - OPT_i = 0$

If algorithm sets x_i to false,

- $B_i - B_{i-1} = f_i$
- $OPT_{i-1} - OPT_i \leq 2t_i$

Case 1: $f_i < 0$ (algorithm sets x_i to true):
$$E[B_i - B_{i-1}] = t_i > 0 = E[OPT_{i-1} - OPT_i]$$

Case 2: $t_i < 0$ (algorithm sets x_i to false):
$$E[B_i - B_{i-1}] = f_i > 0 > 2t_i \geq E[OPT_{i-1} - OPT_i]$$

Relating Algorithm to Optimum

Want to show:

Key Lemma:

$$E[B_i - B_{i-1}] \geq E[OPT_{i-1} - OPT_i]$$

Know:

If algorithm sets x_i to true,

- $B_i - B_{i-1} = t_i$
- $OPT_{i-1} - OPT_i = 0$

alse,

t_i

Equal to

$(t_i - f_i)^2 + 2t_if_i$

Case 3: $t_i \geq 0$, $f_i \geq 0$ (algorithm sets x_i to true w.p. $t_i/_{t_i+f_i}$):

$$E[B_i - B_{i-1}] = t_i \frac{t_i}{t_i+f_i} + f_i \frac{f_i}{t_i+f_i} = \frac{1}{t_i+f_i}(t_i^2 + f_i^2)$$

$$E[OPT_{i-1} - OPT_i] \leq 0\frac{t_i}{t_i+f_i} + 2t_i\frac{f_i}{t_i+f_i} = \frac{1}{t_i+f_i}(2t_if_i)$$

Question

Is there a simple combinatorial <u>deterministic</u> ¾-approximation algorithm?

Deterministic variant?

Greedily maximizing B_i is not good enough:

Clause	Weight
x_1	1
$\bar{x}_1 \vee x_2$	2+ε
x_2	1
$\bar{x}_2 \vee x_3$	2+ε
…..	
x_{n-1}	1
$\bar{x}_{n-1} \vee x_n$	2+ε

Optimal assignment sets all variables to true
OPT = (n-1)(3+ε)

Greedily increasing B_i sets variables $x_1, \dots, x_{n-1}$ to false
GREEDY= (n-1)(2+ε)

A negative result

Poloczek (ESA 11): No deterministic “priority algorithm” can be a ¾ -approximation algorithm, using scheme introduced by Borodin, Nielsen, and Rackoff ‘03.

- Algorithm makes one pass over the variables and sets them.
- Only looks at weights of clauses in which current variable appears positively and negatively (not at the other variables in such clauses).
- Restricted in information used to choose next variable to set.

But…

- It is possible…
- … with a two-pass algorithm (Thanks to Ola Svensson).
- First pass: Set variables x_i fractionally (i.e. probability that x_i true), so that $E[W] \geq \frac{3}{4}\ OPT$.
- Second pass: Use method of conditional expectations to get deterministic solution of value at least as much.

Buchbinder et al.'s approach

- Keep two bounds expected fractional solution
  - **Lower bound LB** = weight of clauses already satisfied
  - **Upper bound UB** = weight of clauses not yet unsatisfied

expected

- Greedy can focus on two things:
  - maximize **LB**,
  - maximize **UB**,

but either choice has bad examp

expected

- Key idea: make choices to increase **B =** ½ (**LB**+**UB**)

As before

Let t_i be (expected) increase in bound B_{i-1} if we set x_i true; f_i be (expected) increase in bound if we set x_i false.

<u>Algorithm</u>:

For $i \leftarrow 1$ to n

- if $t_i < 0$, set x_i to 0
- if $f_i < 0$, set x_i to 1
- else, set x_i to $\frac{t_i}{t_i + f_i}$

For $i \leftarrow 1$ to n

- If $E[W | X_{i-1}, x_i \leftarrow true] \geq E[W | X_{i-1}, x_i \leftarrow false]$, set x_i true
- Else set x_i false

Analysis

- Proof that after the first pass $E[W] \geq \frac{3}{4}\ OPT$ is almost the same as before.
- Proof that final solution output has value at least $E[W] \geq \frac{3}{4}\ OPT$ is via method of conditional expectation.
- Algorithm can be implemented in linear time.

Experimental Analysis

- How well do these algorithms work on structured instances?
- How do they compare to other types of algorithms (e.g. local search)?
- Can we use the randomization to our advantage?

The Instances

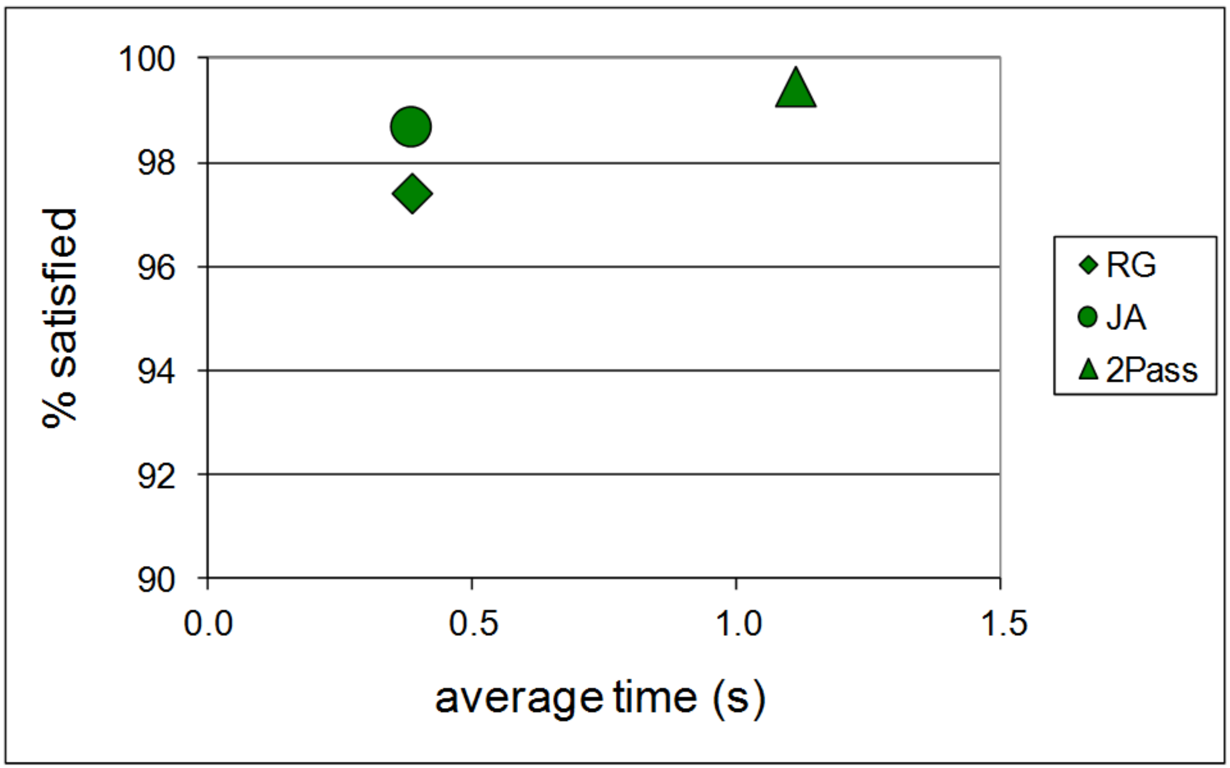
- From SAT and MAX SAT competitions in 2014 and 2015, all unweighted:
  - Industrial/applications: formal verification, crypto attacks, etc (300 + 55 instances)
  - Crafted: Max cut, graph isomorphism, etc (300 + 402 instances)
  - Random: With various ratios of clauses/variables (225 + 702 instances)
- Sizes:
  - Average for industrial: .5M variables in 2M clauses
  - Largest: 14M in 53M clauses
  - Larger in SAT instances than MAX SAT

The Measure

- Rather than approximation ratio, we use the *totality ratio*, ratio of satisfied clauses to the number of clauses in the input.

Greedy Algorithms

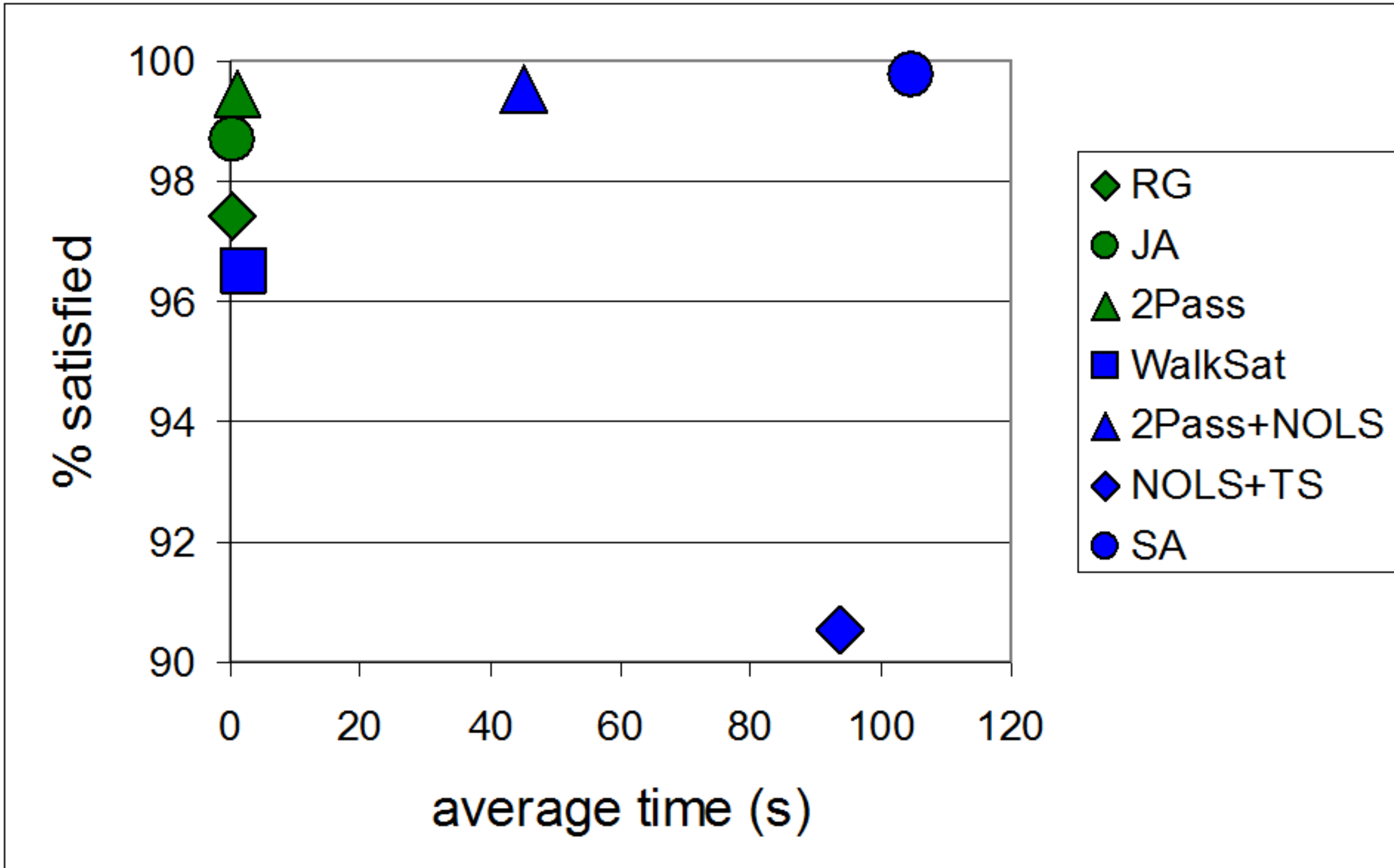
SAT/Industrial instances: Johnson's algorithm (JA) versus Randomized Greedy (RG) versus the 2-pass algorithm (2Pass).



Local Search

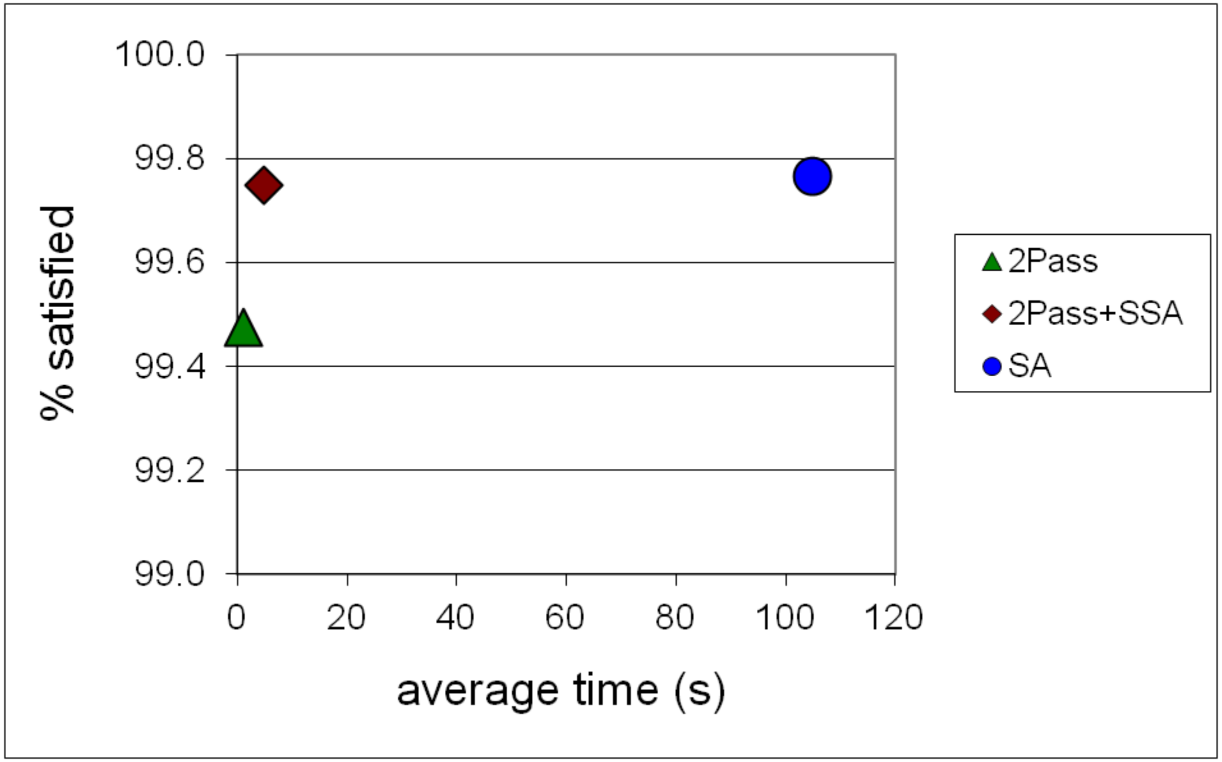
We compared the greedy algorithms versus a number of local search algorithms studied by Pankratov and Borodin (SAT 2010).

- WalkSAT: Selman, Kautz, Cohen (1993), Kautz (2014)
- Non-Oblivious Local Search (NOLS): Khanna, Motwani, Sudan, Vazirani (1998)
- Simulated Annealing (SA): Spears (1993)


100
98
96
94
92
90
% satisfied
0
20
40
60
80
100
120
average time (s)
RG
JA
2Pass
WalkSat
2Pass+NOLS
NOLS+TS
SA

A Hybrid Algorithm

Adding the last 10 iterations of simulated annealing on top of 2-Pass worked really well, not that much slower. The last 10 iterations by themselves was slightly faster, only slightly worse.



Randomization

- Suppose we randomize over the variable orderings? Costello, Shapira, and Tetali (SODA 11) show this improves the worst-case performance of Johnson’s algorithm.
- For industrial instances, this makes the performance of the greedy algorithms worse: Johnson’s alg from 98% to 95.8%, RG from 95.7% to 92.8%.

Randomization

- What about multiple trials of RG (10x)?
- Increases average fraction of satisfied clause by only 0.07%.

Conclusion

- We show this two-pass idea works for other problems as well (e.g. deterministic ½-approximation algorithm for MAX DICUT, MAX NAE SAT).
- Can we characterize the problems for which it does work?

Conclusion

- More broadly, are there other places in which we can reduce the computation needed for approximation algorithms and make them practical?
  - E.g. Trevisan 13/Soto 15 give a .614-approximation algorithm for Max Cut using a spectral algorithm.
  - Can we beat ¾ using a spectral algorithm?
    - For just MAX 2SAT?
    - We can get .817 for *balanced* instances (Paul, Poloczek, W LATIN 16)
    - Curiously, the algorithm seems to beat the GW SDP algorithm on average in practice (Paul et al.)

Thanks for your time and attention.