The two stripe symmetric circulant TSP is in P

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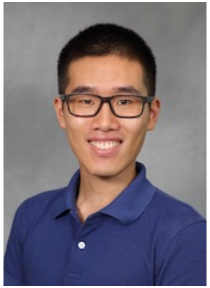
Joint with

Billy Jin



and

Sam Gutekunst

Slides due to Billy

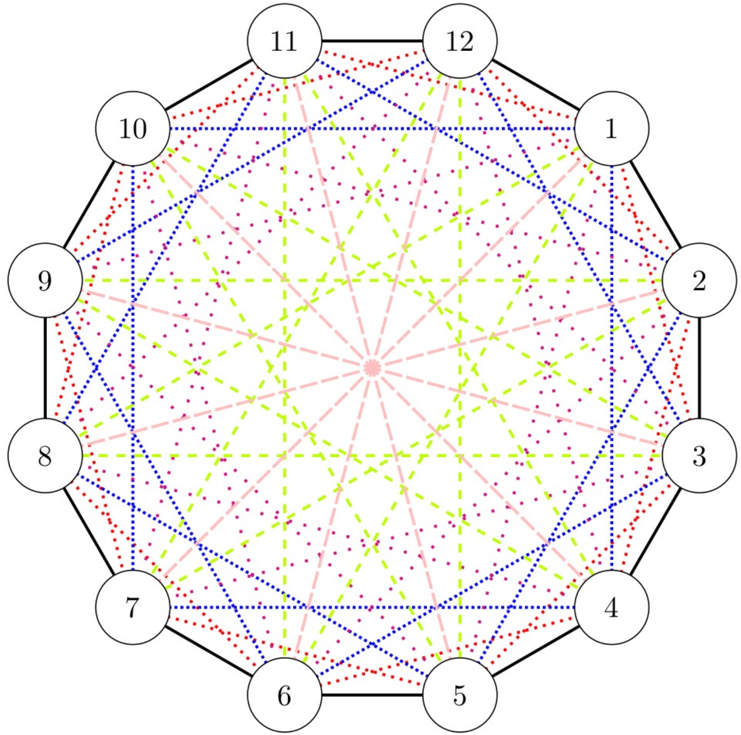
What is the circulant TSP?

- TSP where the matrix of edge costs is <u>circulant</u>
  - Vertex set $V = \{0, 1, \dots, n-1\}$.
  - Costs $c_1, c_2, \dots, c_{\lfloor \frac{n}{2} \rfloor}$
  - All edges $\{i, j\}$ with $i - j \equiv k \pmod{n}$ or $j - i \equiv k \pmod{n}$ have the same cost c_k.

$$\begin{pmatrix} 0 & c_1 & c_2 & \cdots & & \\ c_1 & 0 & c_1 & c_2 & \cdots & \\ c_2 & c_1 & 0 & c_1 & c_2 & \cdots \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

- Do not assume edge costs satisfy triangle inequality

Circulant Symmetry



Why study the circulant TSP?

Some problems become easy when restricted to circulant instances:

- Decision problem of if graph is Hamiltonian [Burkhard, Sandholzer '80]
- Min-cost Hamiltonian path [Bach, Luby, Goldwasser]

Others remain NP-hard:

- Max clique
- Graph colouring

[Codenotti, Gerace, Vigna '98]

<u>Open Problem:</u> Is there a polynomial-time algorithm for circulant TSP?

[Burkhard '98]

[Burkhard, Deĭneko, van Dal, van der Veen, Woeginger '98]

[Lawler, Lenstra, Rinnooy Kan, Shmoys '85]

This talk: 2-stripe symmetric circulant TSP

- Special case of circulant TSP where only 2 of the edge costs $c_1, c_2, \ldots, c_{\lfloor n/2 \rfloor}$ are finite.

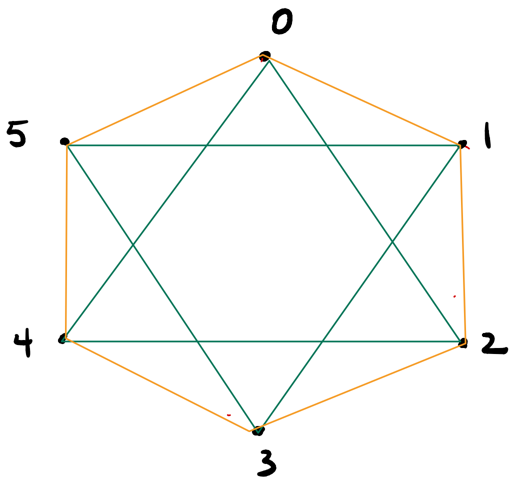
- Let a_1, a_2 be edge lengths of finite cost, so that

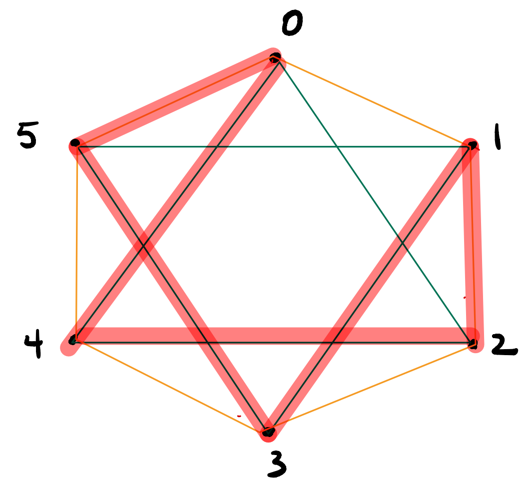
$$c_{a_1} \leq c_{a_2} < +\infty, \quad \text{and } c_i = \infty \text{ for } i \notin \{a_1, a_2\}.$$

- <u>Goal</u>: Find a Ham. cycle minimizing # edges of length a_2 used.

- WLOG, assume $c_{a_1} = 0$ and $c_{a_2} = 1$.

Circulant symmetry with 2 stripes





$n = 6$

$a_1 = 2$

$a_2 = 1$

Previous work on 2-stripe circulant TSP

- [Greco, Gerace '07] [Gerace, Greco '08] initiated the study of 2-stripe circulant TSP and gave some partial results

- [Yang, Burkhard, Çela, Woeginger '97] gave a polynomial-time algorithm for asymmetric 2-stripe circulant TSP

Main Results

1. A structural characterization of the optimal tour. ← this talk

2. A polynomial-time* algorithm to find an optimal tour.

Technical aside: What is "polynomial time"?

- Input is $n, a_1, a_2 \rightsquigarrow O(\log n)$ bits.

- So cannot output entire tour in "polynomial time". Instead:
  - We can characterize the optimal tour using a small number of parameters
  - We can calculate these parameters in $O(\log^2 n)$ time
  - Given these parameters, we can easily describe the optimal tour

- In particular, <u>decision problem</u> is in P.

Outline of remainder of talk

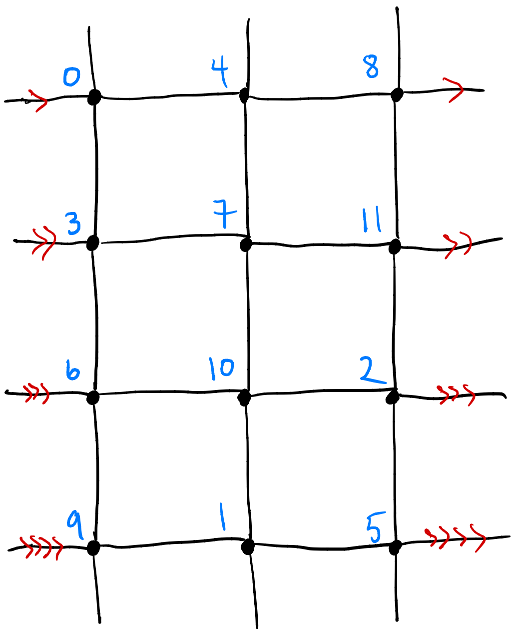
1. Drawing an instance

2. Easy cases

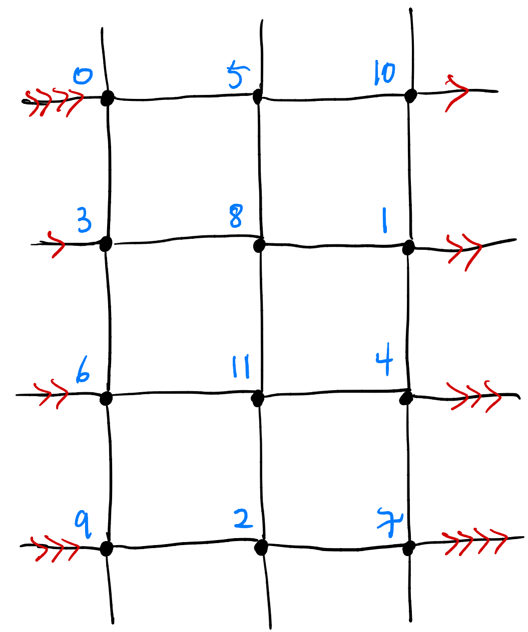
3. Characterization of an optimal tour

4. Proof sketch

Drawing an instance



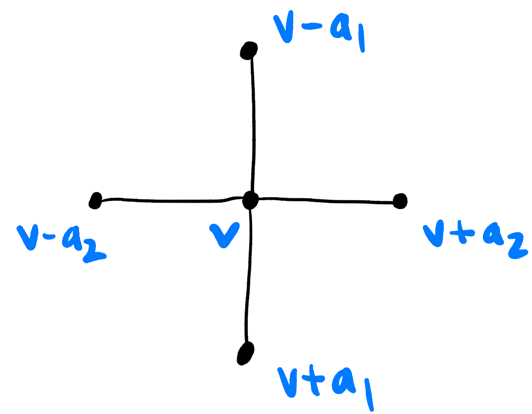
$n = 12, \; a_1 = 3, \; a_2 = 4$



$n = 12, \; a_1 = 3, \; a_2 = 5$

Drawing an instance

- $c = \gcd(n, a_1)$ columns
- $r = \dfrac{n}{\gcd(n,a_1)}$ rows
- vertical edges are cheap / good
- horizontal edges are expensive / bad

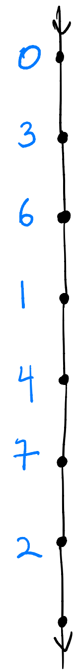


Note: Throughout this talk, vertex labels will be taken (mod n).

- e.g. $n=6$: $\ldots -3, 3, 9, \ldots$ all refer to the same vertex.

Easy cases

1. If $\gcd(n, a_1) = 1$, cost of optimal tour is 0.

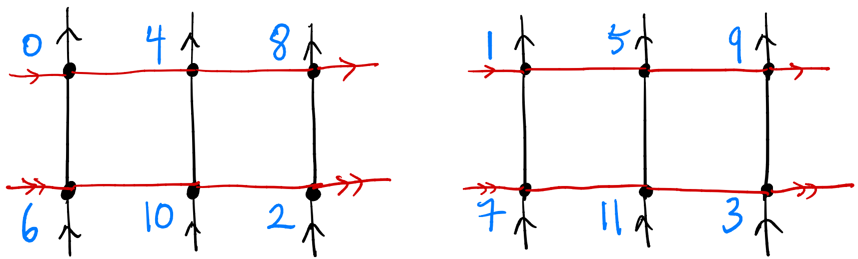


$n = 8$

$a_1 = 3$

Easy cases

2. If $\gcd(n, a_1, a_2) > 1$, cost of optimal tour is ∞.

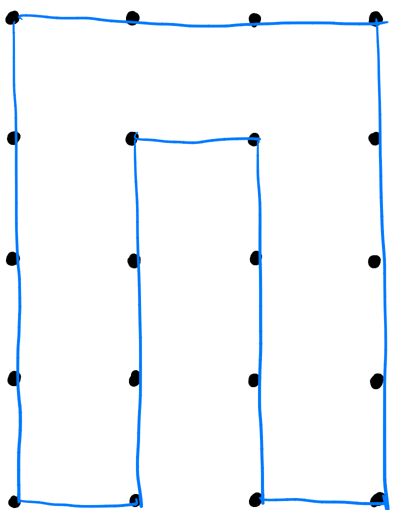


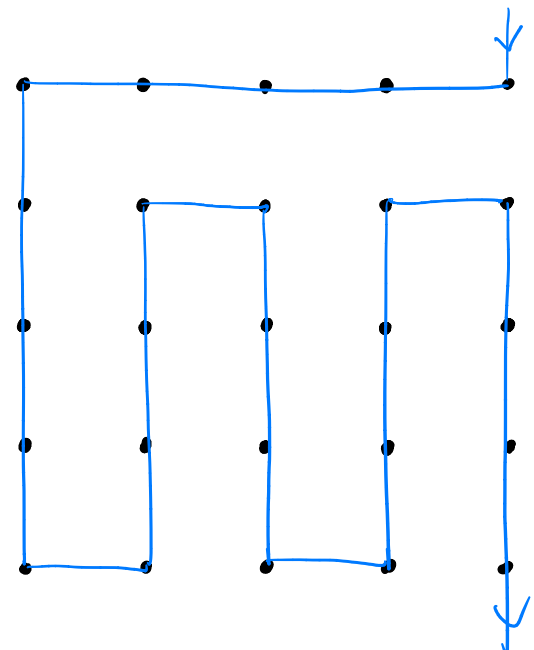
$n = 12, \quad a_1 = 6, \quad a_2 = 4$

An upper bound

<u>Lemma</u>. $OPT \le 2(c-1)$.

<u>Proof</u> (by picture).



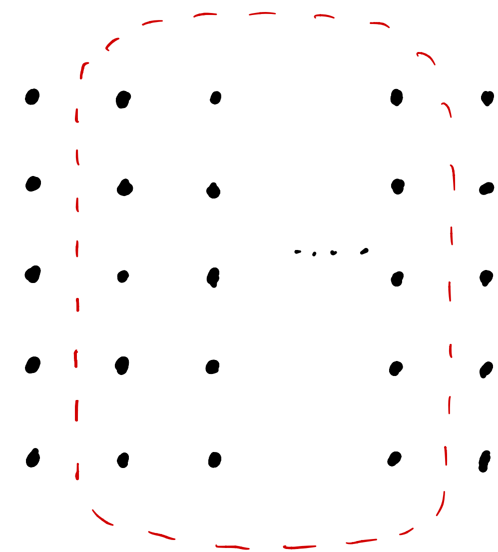


A lower bound

<u>Lemma</u>. $OPT \geq c$.

<u>Proof</u>. If $OPT < c$, there is a pair of adjacent columns with no horizontal edge between them.

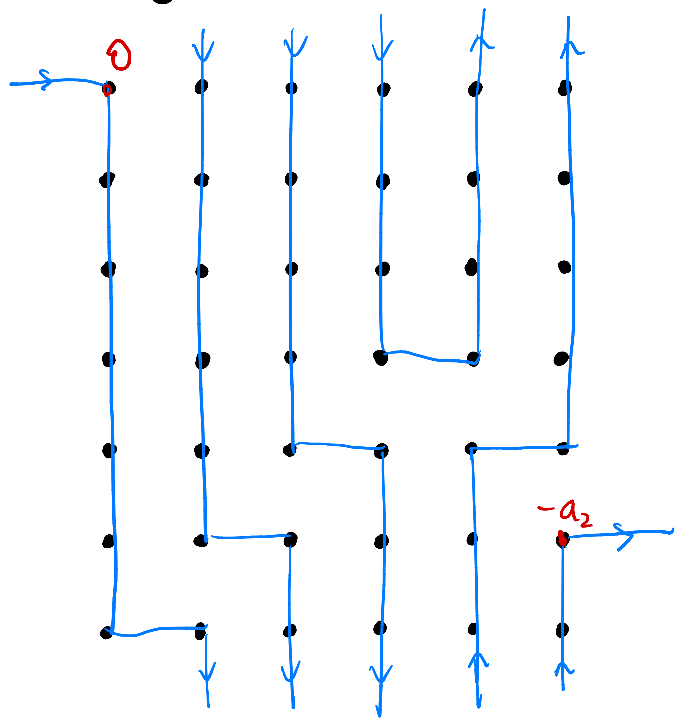
This implies the tour must use ≥ 2 edges between every other pair of adjacent columns.



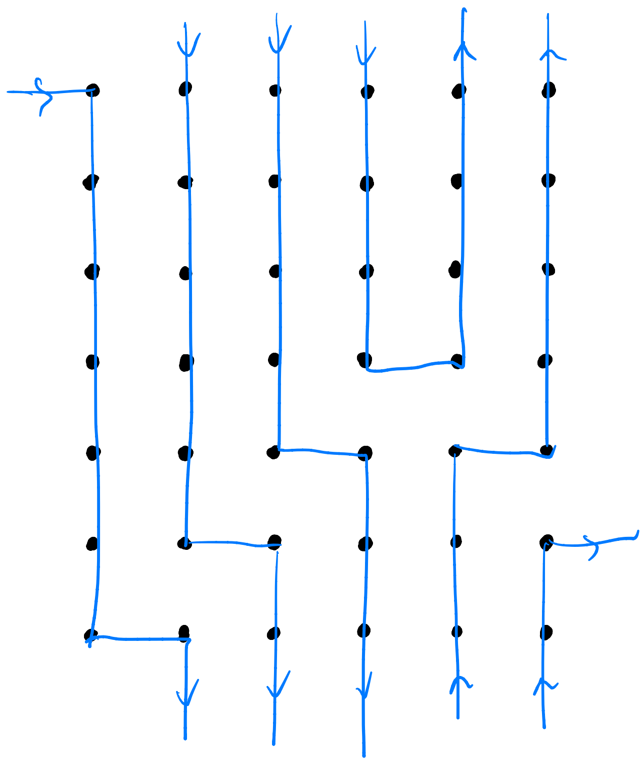
$\Rightarrow$ cost $\geq 2(c-1)$. $(\Rightarrow\Leftarrow)$

When is lower bound achieved?

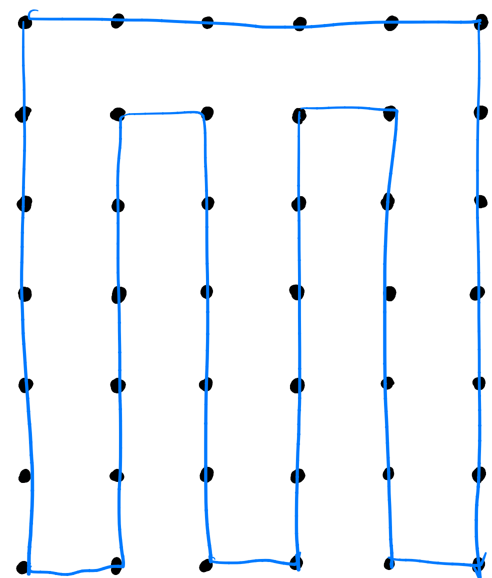
$\mathrm{cost}(T) = c \iff T$ uses 1 bad edge between each pair of adjacent columns.



$$c \leq OPT \leq 2(c-1)$$



"Lower bound tour"



"Upper bound tour"

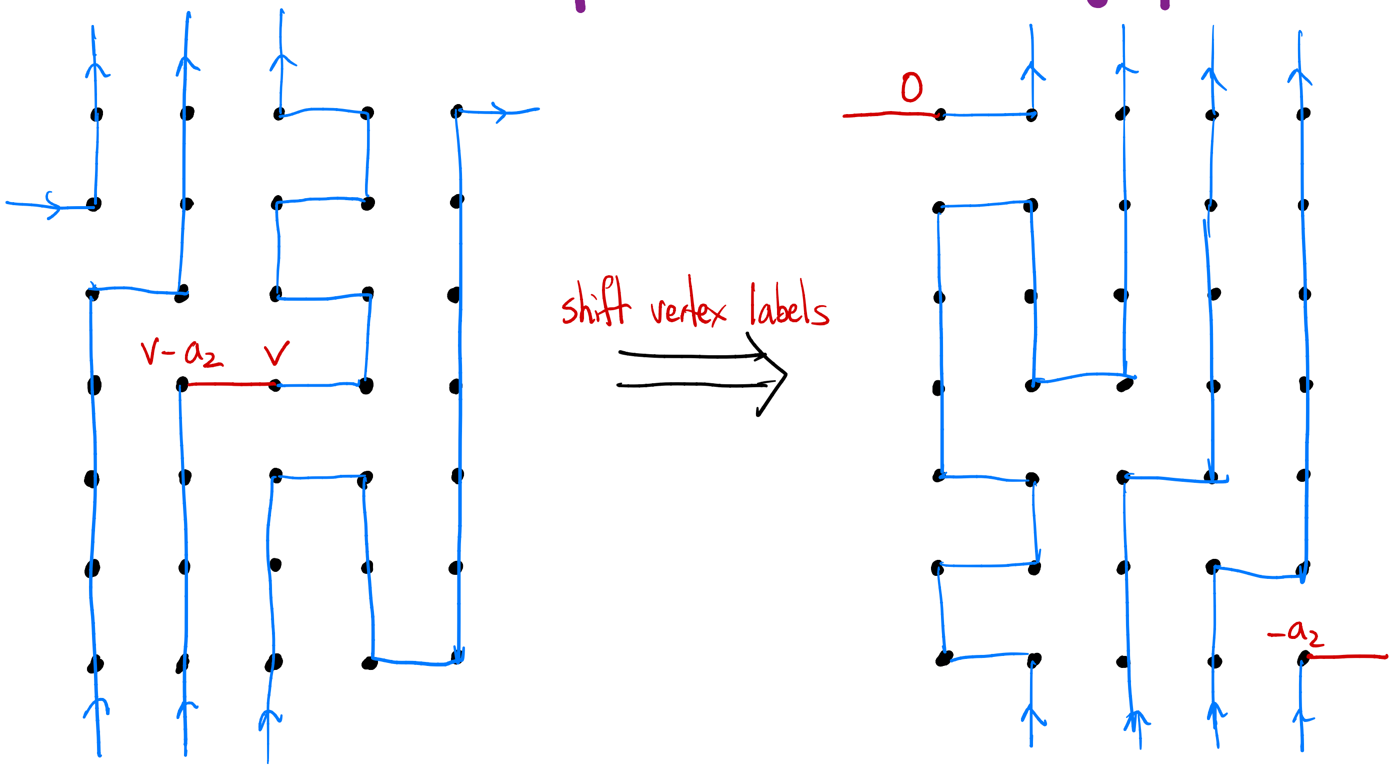
Characterizing an optimal tour

Reduction to Hamiltonian path in a cylinder graph

- Suppose UB tour is not optimal
- Then, opt tour uses < 2 edges between some pair of columns $j, j+1$.
- Cannot use 0 bad edges between $j, j+1$

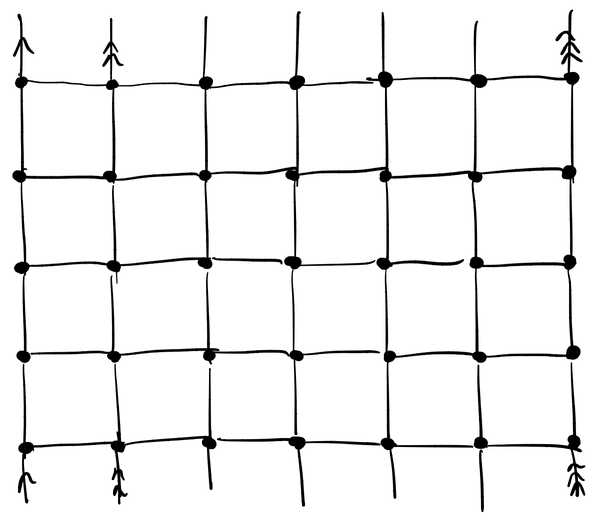
  $\Rightarrow$ uses exactly one bad edge between columns $j, j+1$.
- By shifting vertex labels, assume WLOG that this bad edge is $\{0, -a_2\}$.

Reduction to Hamiltonian path in a cylinder graph



Reduction to Hamiltonian path in a cylinder graph

Definition. An cylinder graph is the same as the graphs we have been drawing before, except with no wraparound edges between the first and last column.



Cylinder graph with $r=5$, $c=7$.

Reduction to Hamiltonian path in a cylinder graph

Lemma. An optimal tour is either:

1. The upper-bound tour (which costs $2(c-1)$), or

2. A min-cost Hamiltonian path from 0 to $-a_2$ in cylinder graph, plus the wraparound edge $\{-a_2, 0\}$.

GG paths*

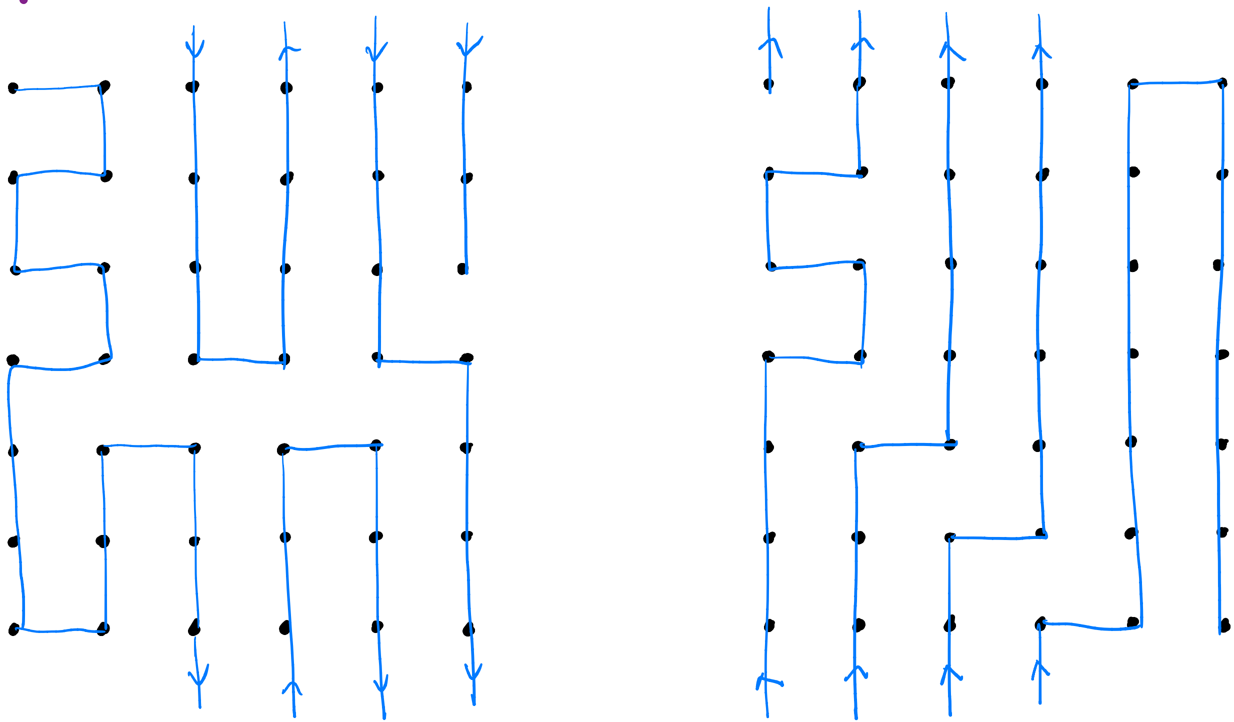
What do min-cost Hamiltonian paths from the first column to the last column of a cylinder graph look like?

Definition. A GG path in a cylinder graph is a Hamiltonian path that:

1. Starts at 0 and ends in the last column,
2. Uses exactly 1 bad edge between each pair of consecutive columns from the second onward.

* Named after Gerace and Greco.

GG paths: Illustration



"In a GG path, all the funky business is between the first 2 columns."

Main structural theorem

<u>Theorem 1</u>. Let v be any vertex in the last column of a cylinder graph. If a Hamiltonian path from 0 to v exists, then there is always a cheapest one that is a GG path.

Structure of the optimal tour

Recall:

__Lemma.__ An optimal tour is either:

1. The upper-bound tour (which costs $2(c-1)$), or
2. A min-cost Hamiltonian path from 0 to $-a_2$ in cylinder graph, plus the wraparound edge $\{0, -a_2\}$.

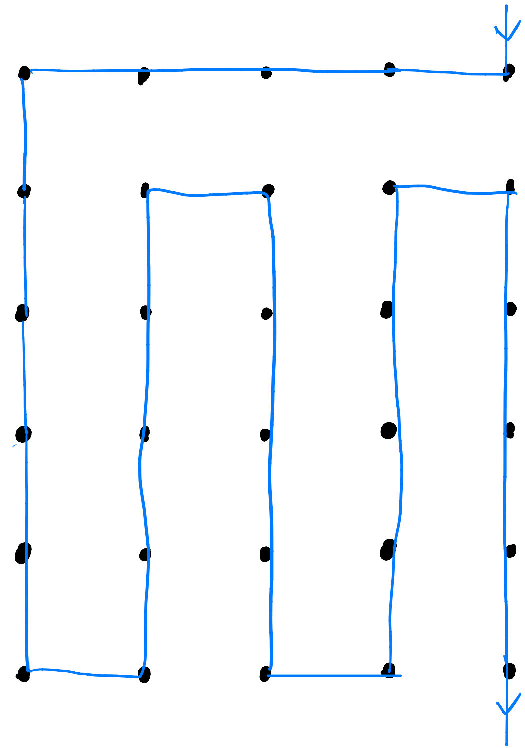
Structure of the optimal tour

Theorem 1 implies the following characterization of the optimal tour.

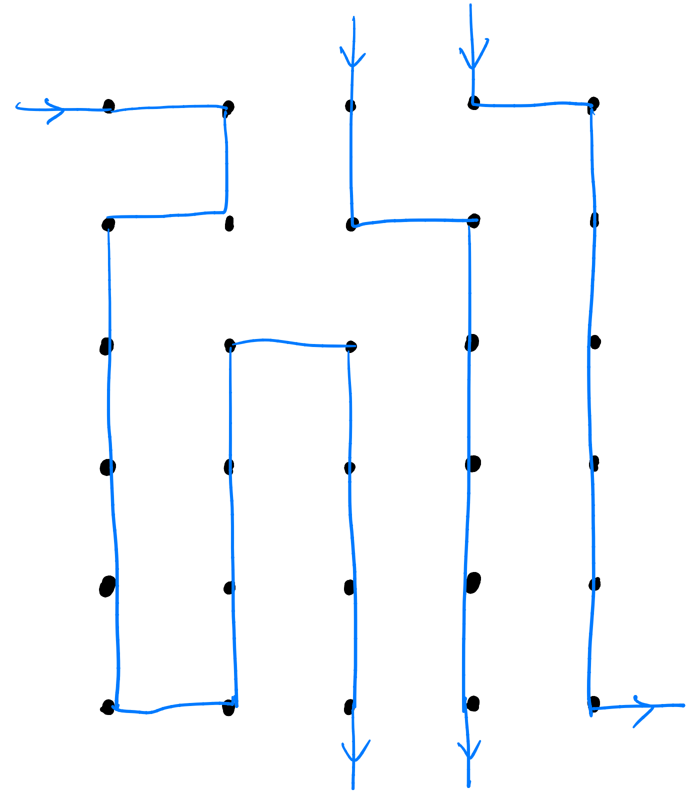
Corollary. In 2-stripe circulant TSP, an optimal tour is either

1. The upper bound tour, with cost $2(c-1)$, or
2. A cheapest GG-path from 0 to $-a_2$, with the wraparound edge from $-a_2$ to 0 appended.

Optimal tour is either . . .



. . . upper-bound tour, or



. . . 66 path plus wraparound edge.

Proof of main structural theorem

Theorem 1. Let v be any vertex in the last column of a cylinder graph. If a Hamiltonian path from 0 to v exists, then there is always a cheapest one that is a GG path.

Proof of main structural theorem

Consider a counterexample to Theorem 1. Consists of

1. An $r \times c$ cylinder graph G,
2. A vertex v in the last column such that the min-cost Hamiltonian path from 0 to v is cheaper than any 0-v GG path.

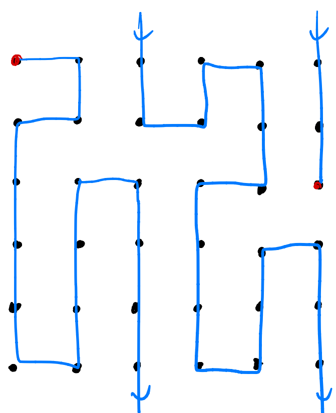
Let P be a min-cost Hamiltonian path from 0 to v.

Minimal counterexample

Among all counterexamples, choose one such that

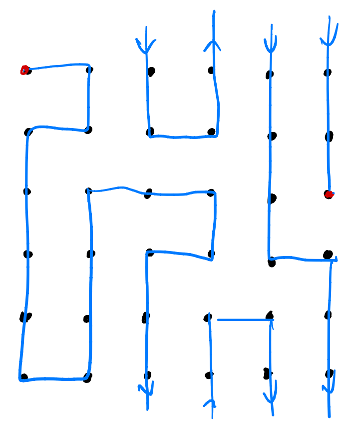
1. $r+c$ is minimal,

2. The reverse lexicographic order of the bad edges in P is minimal.



lex. order $= (3, 1, 1, 3, 1)$

rev. lex. order $= (1, 3, 1, 1, 3)$



lex. order $= (3, 1, 3, 1, 1)$

rev. lex. order $= (1, 1, 3, 1, 3)$

Why reverse lexicographic order?

Among all Hamiltonian paths that use m bad edges,

G_G paths minimize reverse lexicographic order.

Applying transformations

- Recall: minimal counterexample $(r, c, 0\text{-}v$ Ham. path $P)$
- We will begin applying transformations to P
- These transformations generate a sequence of subgraphs:

$$P = H_0 \to H_1 \to H_2 \to \cdots \to H_k$$

- These subgraphs satisfy, $\forall i \geq 1$:

$$\text{cost}(H_i) \leq \text{cost}(P) \quad \text{and} \quad \text{rev. lex.}(H_i) < \text{rev. lex.}(P)$$

- If H_i is a 0-v Hamiltonian path, we are done!

Applying transformations

- We will end up with a sequence of subgraphs:

$$P = H_0 \to H_1 \to H_2 \to \cdots \to H_k$$

- These subgraphs will satisfy

  (i) $\text{cost}(H_i) \leq \text{cost}(P)$

  (ii) rev. lex. order $(H_i) <$ rev. lex. order (P)

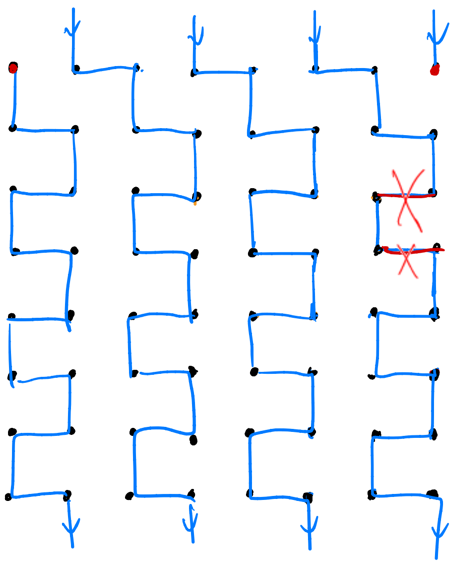
  (iii) H_i is either a 0-v Hamiltonian path, or the disjoint union of a 0-v path and a cycle.

- Can assume each H_i is the disjoint union of a 0-v path P_i and a cycle C_i $\forall i \geq 1$

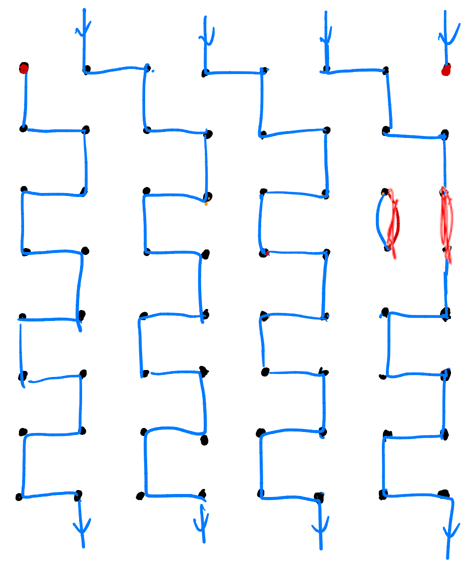
  $\Rightarrow$ lots of structure!

What do transformations look like?

$H_0 \to H_1$ "initial transformation"



$H_0 = P$



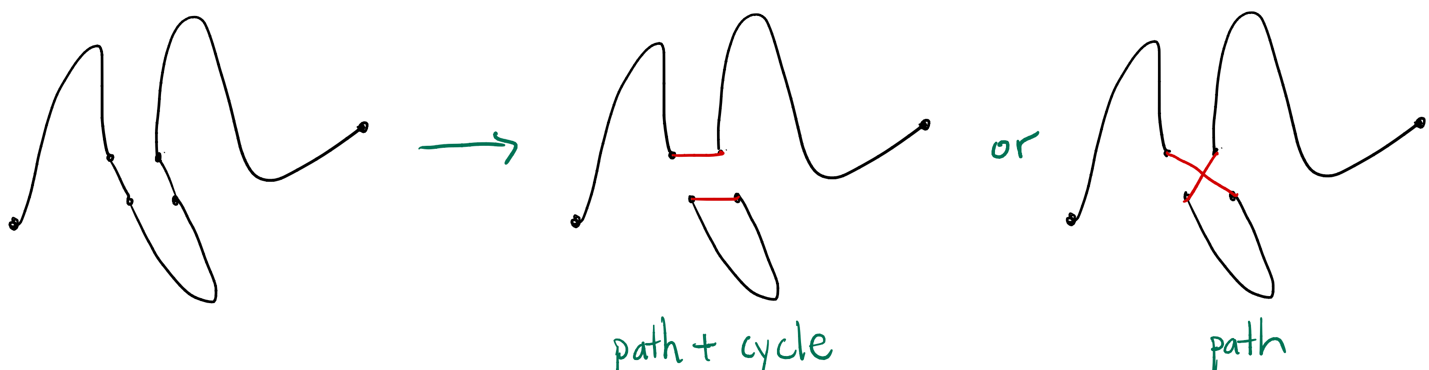
H_1

Why does H_1 satisfy the invariants?

(i) $\mathrm{cost}(H_1) = \mathrm{cost}(P) - 2$

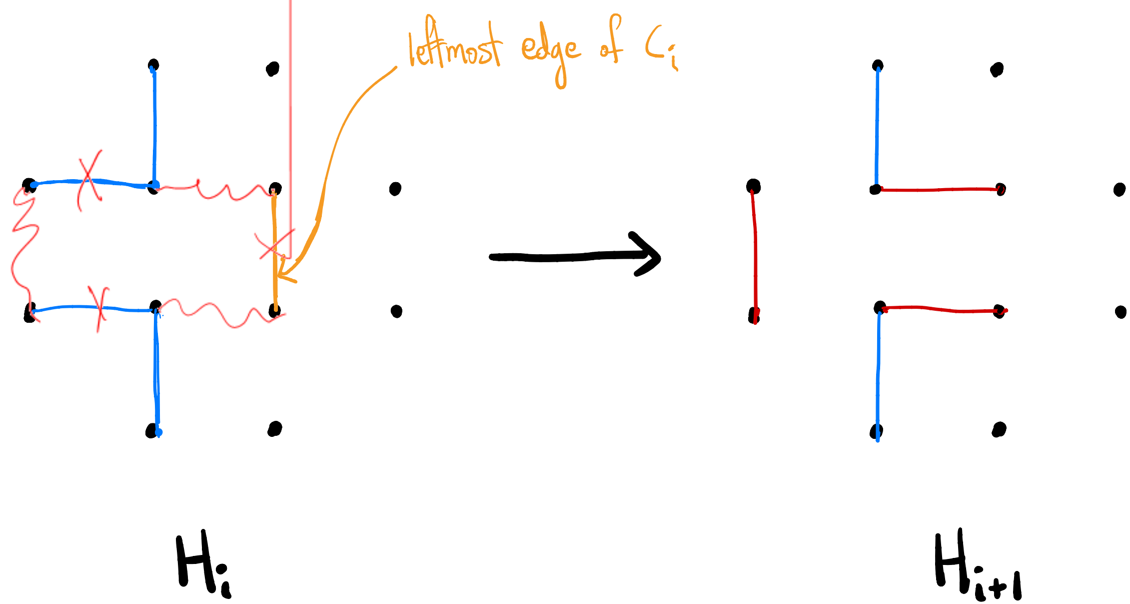
(ii) rev. lex. order $(H_1) <$ rev. lex. order (P)

(iii) H_1 is either a 0-v Hamiltonian path, or the disjoint union of a 0-v path and a cycle.

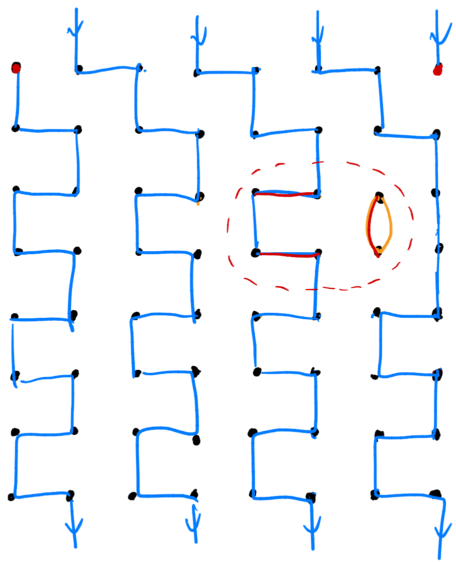


$H_i \to H_{i+1}$ for $i \geq 1$ "cycle propagating transformation"

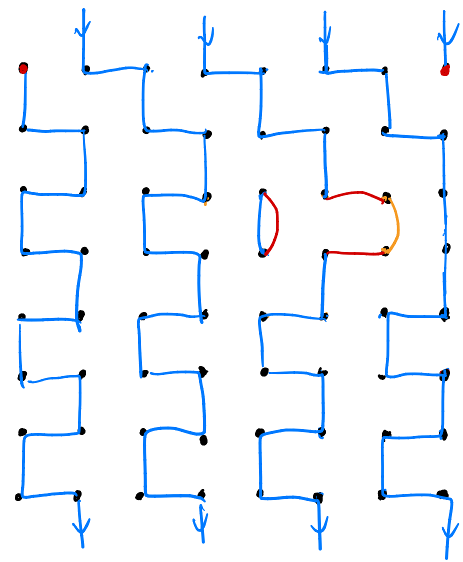
Suppose we are at step i, with subgraph $H_i = P_i \cup C_i$.



Propagating cycles backward



H_i



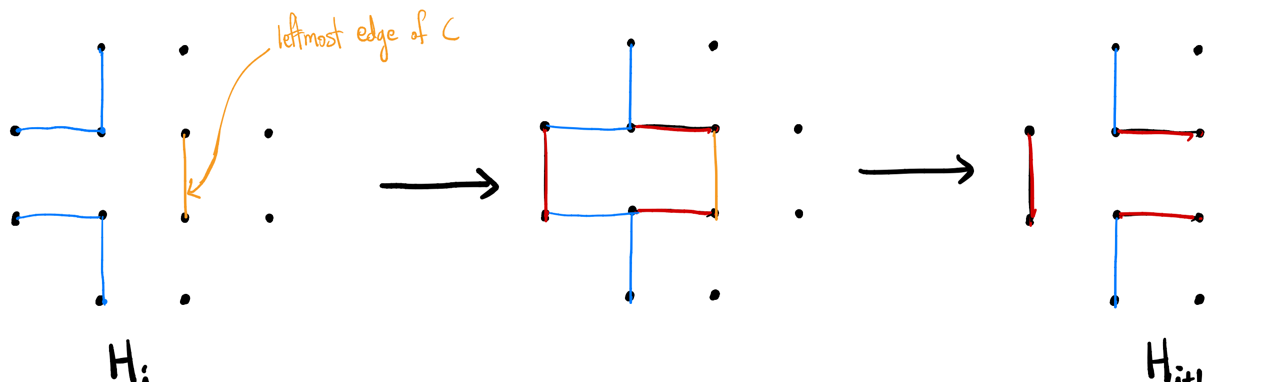
H_{i+1}

Can show that H_{i+1} satisfies invariants

(i) $\text{cost}(H_{i+1}) = \text{cost}(H_i)$

(ii) rev. lex. order(H_{i+1}) < rev. lex. order(P)

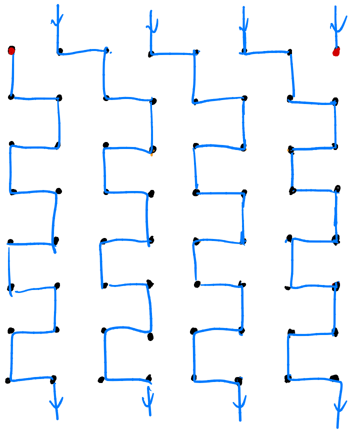
(iii) H_{i+1} is either a 0-v Hamiltonian path, or the disjoint union of a 0-v path and a cycle.



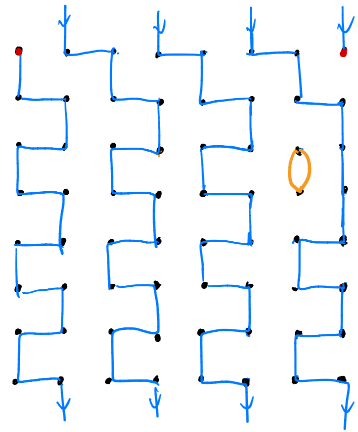
Rinse and repeat

- Apply transformation repeatedly to get subgraphs $H_1, H_2, \ldots, H_k$, where
  - $H_i = P_i \cup C_i$
  - The leftmost vertical edge of C_{i+1} is further to the left than the leftmost vertical edge of C_i

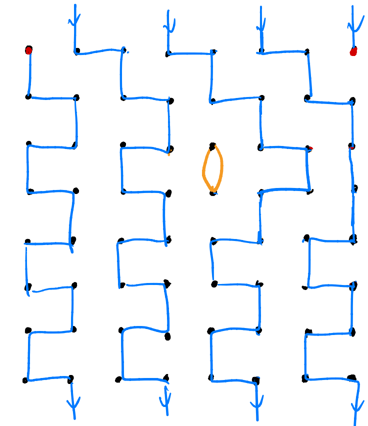
$\therefore$ The process terminates and $k \leq c$.



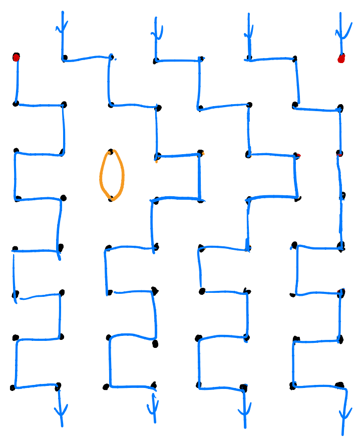
$H_0 = P$



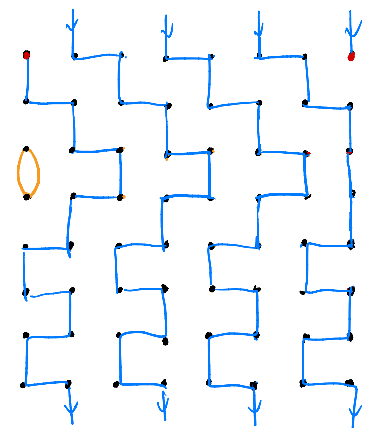
H_1



H_2



H_3



H_4

Sketch of rest of the proof

Three steps:

1. Show that C_k is a 2-cycle in the first column.

2. Show that $C_1, \ldots, C_k$ are all 2-cycles

3. Use 2. and minimal counterexample to get a contradiction.

Conclusion

We gave a characterization of optimal tour in symmetric 2-stripe circulant TSP. Opt. tour is either

1. The upper-bound tour, which costs $2(c-1)$, or
2. A cheapest GG path from 0 to $-a_2$, plus the wraparound edge from $-a_2$ to 0.

This can be used to get an efficient algorithm.

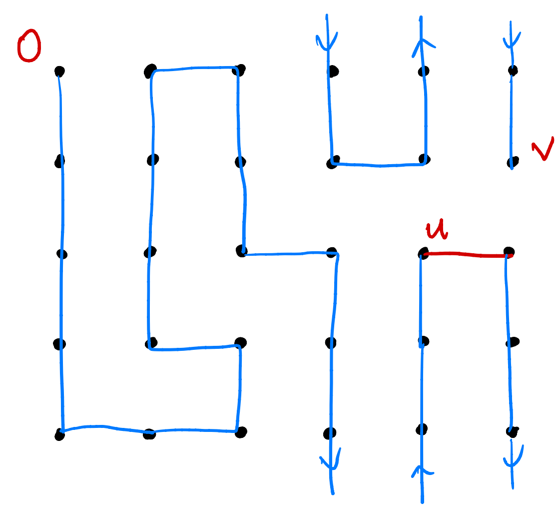
Open Questions

- A polynomial-time algorithm for general circulant TSP?
  - Currently a 2-approximation is known
- As a first step, what about 3-stripe circulant TSP or constant-stripe circulant TSP?

Claim 1. P uses >1 bad edges between the last two columns.

Proof. If not, then P uses $\underline{1}$ bad edge between last two columns.

$\Rightarrow$ Can reduce to a graph with one fewer column.



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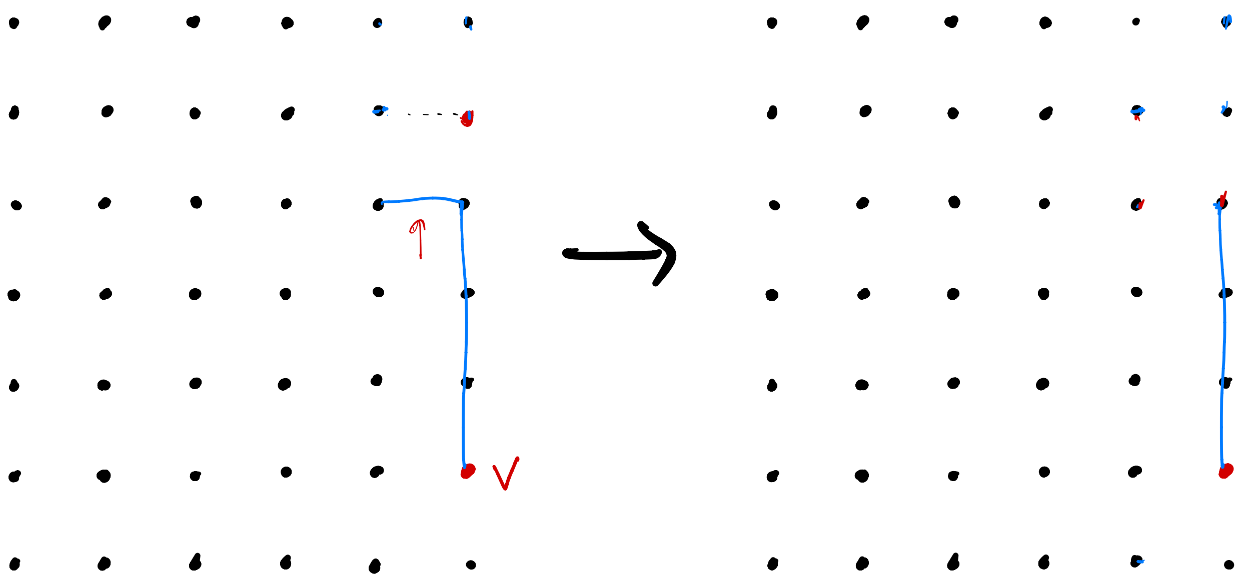

$P = H_0$ $\qquad\qquad$ H_1

- If H_1 were a path, it would contradict minimality of P.

$\therefore$ H_1 is the disjoint union of a path P_1 and a cycle C_1.

Useful Claim. Let $H = P \cup C$ be the disjoint union of a path + cycle.

Let $x_1x_2 \in P$, $y_1y_2 \in C$. Then $H - \{x_1x_2, y_1y_2\} + \{x_1y_1, x_2y_2\}$ is a path.

Pf.