Some Open Problems in Approximation Algorithms

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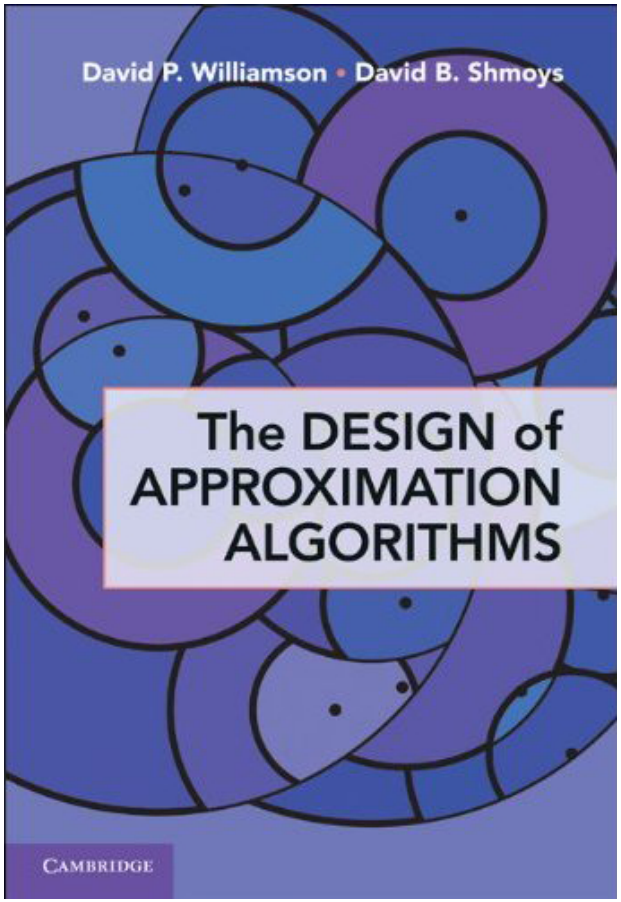
The book



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Outline

- Introduction
- A brief history and some early results
- Ten open problems
- Some concluding thoughts and issues

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An old engineering slogan: "Fast. Cheap. Reliable. Choose any two."

Similarly, if P $\neq$ NP, then for any NP-hard problem, choose two:

- a polynomial-time algorithm
- that for every instance
- finds the optimal solution.

All work on these problems relaxes at least one of these conditions.

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3. Drop the optimality requirement: heuristics, metaheuristics, . . .
   - Pro: Often we only need a solution that is "good enough"
   - Con: How good is the solution we get?

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Definition

An α-approximation algorithm for a discrete optimization problem Π, for any instance of Π, runs in polynomial time and produces a solution of cost within α times the cost of an optimal solution to the instance.

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- Randomized approximation algorithms: expected value is within α of optimal.
- Additive approximation algorithms: solution is within additive error of optimal.
- Polynomial-time approximation scheme (PTAS): for any $\epsilon > 0$, a $(1 + \epsilon)$-approximation algorithm.

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Johnson's 1974 paper gives:

- $(\ln n)$-approximation algorithm for (unweighted) set cover problem (see also Lovász (1975))
- $1/2$-approximation algorithm for maximum satisfiability problem
- approximation algorithms for vertex coloring and maximum clique problem

Johnson's paper

Paper ends with:

> *The results described in this paper indicate a possible classification of optimization problems as to the behavior of their approximation algorithms. Such a classification must remain tentative, at least until the existence of polynomial-time algorithms for finding optimal solutions has been proved or disproved. In the meantime, many questions can be asked. Are there indeed* $O(\log n)$ *coloring algorithms? Are there any clique finding algorithms better than* $O(n^{\epsilon})$ *for all* $\epsilon > 0$*? Where do other optimization problems fit into the scheme of things? What is it that makes algorithms for different problems behave in the same way? Is there some stronger kind of reducibility than the simple polynomial reducibility that will explain these results, or are they due to some structural similarity between the problems as we define them? And what other types of behavior and ways of analyzing and measuring it are possible?*

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The community spent the next few decades trying to answer some of these questions.

Some Early, Easy Results

Maximum Satisfiability Problem

Input:

- n boolean variables $x_1, x_2, \ldots, x_n$
- m clauses of disjunctions of variables or negations, e.g. $x_1 \vee \bar{x}_5 \vee x_{12}$.
- clause weights $w_j \geq 0$ for $1 \leq j \leq m$.

Goal: Find a setting of the x_i that maximizes the weight of satisfied clauses.

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An easy algorithm: Set each x_i true with probability 1/2.

Some Early, Easy Results

Maximum Satisfiability Problem

$$\begin{aligned}\text{Expected weight of satisfied clauses} &= \sum_{j=1}^{m} w_j \cdot \Pr[\text{Clause } j \text{ satisfied}] \\ &\geq \frac{1}{2} \sum_{j=1}^{m} w_j \\ &\geq \frac{1}{2} \text{OPT}\,.\end{aligned}$$

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If each clause has exactly three literals (MAX E3SAT), can show a $\frac{7}{8}$-approximation algorithm (Johnson 1974).

Some Easy, Early Results

Vertex Cover Problem

Input:

- Undirected graph $G = (V, E)$
- weights $w_v \geq 0$ for all $v \in V$

Goal: Find $S \subseteq V$ of minimum weight so that each edge has at least one endpoint in S.

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Goal: Find $S \subseteq V$ of minimum weight so that each edge has at least one endpoint in S.

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} w_v x_v \\ \text{subject to} \quad & x_u + x_v \geq 1, \qquad \forall (u, v) \in E, \\ & x_v \geq 0, \qquad \forall v \in V. \end{aligned}$$

An easy algorithm (Hochbaum 1982): Solve LP, let $S = \{v \in V : x_v^* \geq 1/2\}$.

Some Easy, Early Results

Algorithm: Solve LP, let $S = \{v \in V : x_v^* \geq 1/2\}$.

Clearly $\sum_{v \in S} w_j \leq 2 \sum_{v \in V} w_v x_v^* \leq 2\,\mathrm{OPT}$.

Also, since $x_u^* + x_v^* \geq 1$ for all $(u, v) \in V$, either $u \in S$ or $v \in S$ (or both).

Thus, this is a 2-approximation algorithm.

Some Easy, Early Results

Consider also the dual of LP relaxation:

$$\begin{aligned} \text{maximize} \quad & \sum_{(u,v)\in E} y_{(u,v)} \\ \text{subject to} \quad & \sum_{u:(u,v)\in E} y_{(u,v)} \leq w_v, \qquad \forall v \in V, \\ & y_{(u,v)} \geq 0, \qquad \forall (u,v) \in E. \end{aligned}$$

A *primal-dual* algorithm (Bar-Yehuda, Even 1981):

- Start with $S = \emptyset$, $y = 0$
- While S is not a vertex cover since $(u, v) \in E$ uncovered
  - Increase $y_{(u,v)}$ until $\sum_{a:(a,b)\in E} y_{(a,b)} = w_b$ for some $b \in V$
  - Add b to S

Some Easy, Early Results

$$\begin{aligned}
\text{maximize} \quad & \sum_{(u,v)\in E} y_{(u,v)} \\
\text{subject to} \quad & \sum_{u:(u,v)\in E} y_{(u,v)} \leq w_v, \qquad \forall v \in V, \\
& y_{(u,v)} \geq 0, \qquad \forall (u,v) \in E.
\end{aligned}$$

This algorithm is also a 2-approximation algorithm since

$$\begin{aligned}
\sum_{v\in S} w_v &= \sum_{v\in S} \sum_{u:(u,v)\in E} y_{(u,v)} \\
&\leq 2 \sum_{(u,v)\in E} y_{(u,v)} \leq 2\,\mathrm{OPT}
\end{aligned}$$

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Approximate Min-Max Theorems

Most (not all!) approximation algorithms are approximate min-max theorems:

For minimization problems, we have some polytime computable bound R such that

$$R \leq \text{OPT} \leq \text{algorithm's soln} \leq \alpha R.$$

For vertex cover, R is the value of the linear programming relaxation.

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For vertex cover, R is the value of the linear programming relaxation.

For maximization problems,

$$R \geq \text{OPT} \geq \text{algorithm's soln} \geq \alpha R.$$

For maximum satisfiability, R is total sum of clause weights.

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Hardness Results

Starting in the 1990s, important progress in showing the nonexistence of approximation algorithms (if P $\neq$ NP).

- The breakthrough: Arora, Lund, Motwani, Sudan, and Szegedy (1998) use a new definition of NP in terms of *probabilistically checkable proofs* to show that no PTAS can exist for a large class of problems unless P $=$ NP.

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- Lund and Yannakakis (1994) show for some $c < 1$, no $(c \ln n)$-approximation algorithm for set cover unless P $=$ NP.
- Feige (1998) improves to show that for *all* $c < 1$, no $(c \ln n)$-approximation for set cover unless NP has $O(n^{\log \log n})$ time algoritms.

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- Håstad (2001) shows that for all $\epsilon > 0$, no $(\frac{7}{8} + \epsilon)$-approximation algorithm for MAX E3SAT unless P $=$ NP.
- Dinur and Safra (2002) show no α-approximation algorithm for vertex cover with $\alpha < 10\sqrt{5} - 21 \approx 1.36$ unless P $=$ NP.

Hardness result: Unique Games

Recent work based on *unique games conjecture*.

Unique Games Problem

Input:

- An undirected graph $G = (V, E)$
- Labels L
- Permutations $\pi_{uv} : L \to L$ for all $(u, v) \in E$

Goal: Find an assignment $\sigma : L \to V$ to maximize the number of satisfied edges; (u, v) satisfied if u labelled with $i \in L$, v labelled with $j \in L$ and $\pi_{uv}(i) = j$.

Conjecture (Unique Games Conjecture (UGC), Khot (2002))

For every $\delta > 0$, there exists some k such that for $|L| = k$ it is NP*-hard to decide whether*

- *At least $(1 - \delta)|E|$ edges are satisfiable*
- *or at most $\delta|E|$ edges are satisfiable*

Hardness results: Unique Games

Some examples:

- Khot and Regev (2008) show that given UGC, there is no α-approximation algorithm for vertex cover with $\alpha < 2$ unless P $=$ NP.

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- Raghavendra (2008), Raghavendra and Steurer (2009) show that the approximability of constraint satisfaction problems tied to integrality gap of semidefinite program, and given UGC, cannot do better than integrality gap unless P $=$ NP.
- Svensson (2010) shows that given a variant of the UGC, there is no α-approximation algorithm for a scheduling problem of Graham with $\alpha < 2$ unless P $=$ NP.

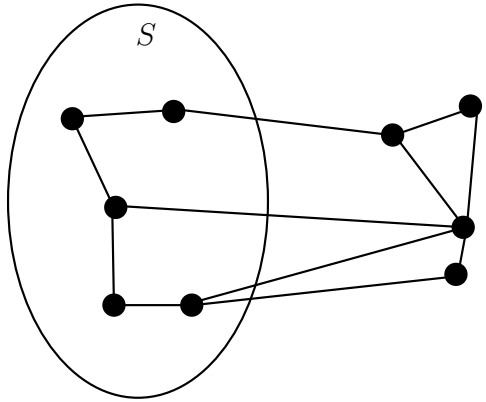
Next: ten open problems from our book.

Problem 10: A primal-dual algorithm for the maximum cut problem

Maximum Cut Problem

Input: An undirected graph $G = (V, E)$ with nonnegative edge weights $w_{ij} \geq 0$ for all $i, j \in V$.

Goal: Find a set of vertices $S \subseteq V$ that maximizes $\sum_{i \in S, j \notin S} w_{ij}$.



Problem 10: A primal-dual algorithm for the maximum cut problem

What's known?

- an $(\alpha - \epsilon)$-approximation algorithm using semidefinite programming (Goemans, W 1995) for

$$\alpha = \min_{-1 \le x \le 1} \frac{\frac{1}{\pi} \arccos(x)}{\frac{1}{2}(1-x)} \approx .87856,$$

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- No β-approximation algorithm possible for constant $\beta > \frac{16}{17} \approx .941$ unless P $=$ NP (Håstad 1997).

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Problem 10: A primal-dual algorithm for the maximum cut problem

The problem:

Solving the semidefinite program is computationally expensive. Can one obtain an $(\alpha - \epsilon)$-approximation algorithm for the problem via computationally easier means? E.g. a primal-dual algorithm?

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A potential start:

(Trevisan, STOC 2009) gives a .531-approximation algorithm via an eigenvalue computation.

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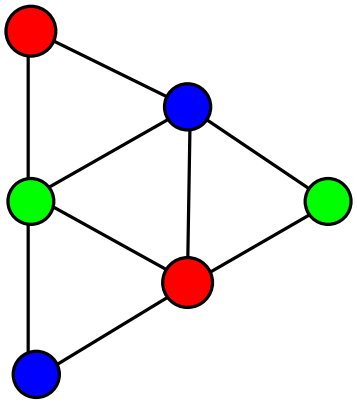
Problem 9: Coloring 3-colorable graphs

Coloring 3-colorable graphs

Input: An undirected, 3-colorable graph $G = (V, E)$.

Goal: Find a k-coloring of the graph with k as small as possible.



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The problem:

Give an algorithm that uses $O(\log n)$ colors for 3-colorable graphs (or show this is not possible modulo some complexity theoretic condition).

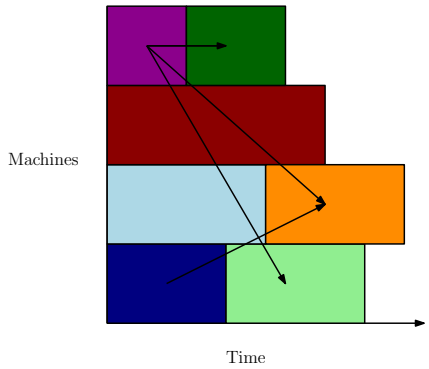
Problem 8: Scheduling related machines with precedence constraints

Scheduling related machines with precedence constraints

Input:

- n jobs with processing requirements $p_1, \ldots, p_n \geq 0$.
- m machines with speeds $s_1 \geq s_2 \geq \cdots \geq s_m > 0$.
- A precedence relation $\prec$ on jobs.

Goal: Find a schedule of minimum length in which all jobs are completely scheduled and if $j \prec j'$, then job j completes before job j' starts. Job j on machine i takes p_j/s_i units of time.



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- If machines are identical, and given a variant of the unique games conjecture, then no α-approximation algorithm is possible for $\alpha < 2$ unless P $=$ NP. (Svensson STOC 2010).

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- If machines are identical, and given a variant of the unique games conjecture, then no α-approximation algorithm is possible for $\alpha < 2$ unless P $=$ NP. (Svensson STOC 2010).

The problem:

Give an α-approximation algorithm for some constant α, or show that $O(\log m)$ is the best possible modulo the unique games conjecture.

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Problem 7: Scheduling unrelated machines

Scheduling unrelated machines

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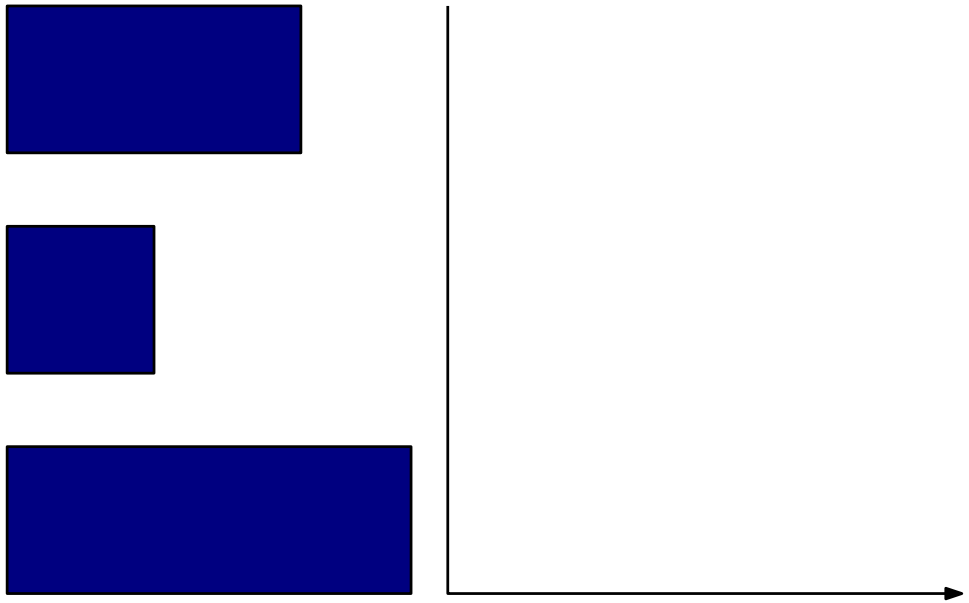
- m machines.
- n jobs with processing requirements p_{ij} for scheduling job j on machine i.

Goal: Find a schedule of minimum length.



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- A 2-approximation algorithm via LP rounding (Lenstra, Shmoys, Tardos 1990)

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- A 2-approximation algorithm via LP rounding (Lenstra, Shmoys, Tardos 1990)
- A 1.94-approximation algorithm if running time is $p_{ij} \in \{p_j, \infty\}$ for all i, j (Svensson STOC 2011).

Problem 7: Scheduling unrelated machines

What's known?

- A 2-approximation algorithm via LP rounding (Lenstra, Shmoys, Tardos 1990)
- A 1.94-approximation algorithm if running time is $p_{ij} \in \{p_j, \infty\}$ for all i, j (Svensson STOC 2011).
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The problem:

Give an α-approximation algorithm for $3/2 \leq \alpha < 2$, or show that this is not possible.

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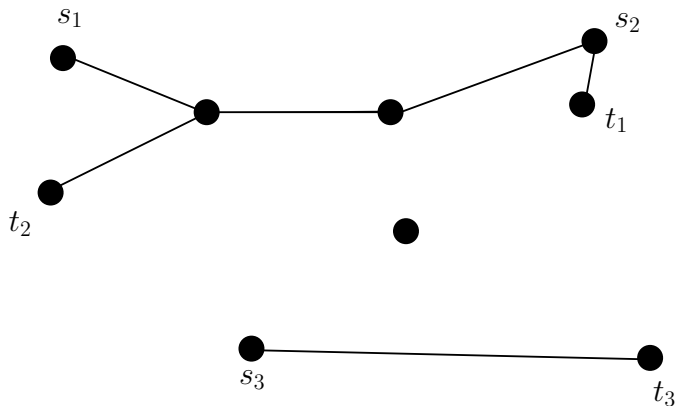
Problem 6: Generalized Steiner tree

Generalized Steiner tree

Input:

- Undirected graph $G = (V, E)$.
- Nonnegative edge costs $c_e \geq 0$ for all $e \in E$.
- k source-sink pairs s_1-t_1, s_2-t_2, ..., s_k-t_k.

Goal: Find edges F of minimum cost so that for each i, s_i and t_i are connected in (V, F).



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The problem

Find an α-approximation algorithm for the generalized Steiner tree problem for constant $\alpha < 2$.

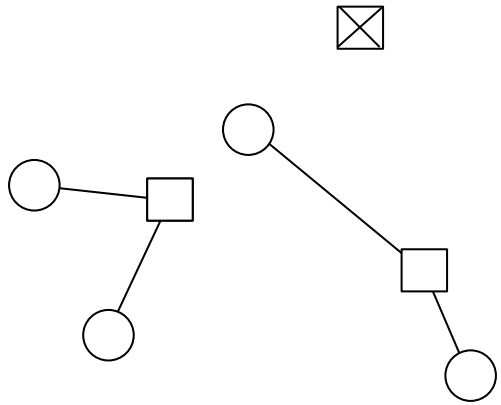
Problem 5: Capacitated facility location

Capacitated facility location

Input:

- A set F of facilities; each $i \in F$ has facility cost $f_i \geq 0$.
- A set D of clients.
- A metric c_{ij} on locations $i, j \in F \cup D$.
- A capacity U on each facility.

Goal: Find $S \subset F$ and assignment $\sigma : D \to S$ such that $|\sigma^{-1}(i)| \leq U$ for all $i \in S$ that minimizes $\sum_{i \in S} f_i + \sum_{j \in D} c_{\sigma(j),j}$.



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Problem 5: Capacitated facility location

What's known?

A local search algorithm: Let S be a set of currently open facilities. As long as it improves the overall cost,

- **Add**: $S \leftarrow S \cup \{i\}$ for $i \notin S$;
- **Drop**: $S \leftarrow S - \{i\}$ for $i \in S$; or
- **Swap**: $S \leftarrow S \cup \{i\} - \{j\}$ for $i \notin S$, $j \in S$.

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- Can show this gives an $(\alpha + \epsilon)$-approximation algorithm for
  - $\alpha = 8$ (Koropolu, Plaxton, Rajaraman 2000)
  - $\alpha = 6$ (Chudak, W 2005)
  - $\alpha = 3$ (Aggarwal et al. 2010)

Problem 5: Capacitated facility location

Let S be local optimal solution, with assignment σ; let S^* be optimal solution with assignment σ^*.
For any $i \in O$, since locally optimal,

$$f_i + \sum_{j \in \sigma^{*-1}(i)} (c_{ij} - c_{\sigma(j),j}) \geq 0.$$

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Summing over all $i \in O$,

$$\sum_{i \in O} f_i + \sum_{j \in D} (c_{\sigma^*(j),j} - c_{\sigma(j),j}) \geq 0,$$

or

$$\sum_{j \in D} c_{\sigma(j),j} \leq \sum_{i \in O} f_i + \sum_{j \in D} c_{\sigma^*(j),j}.$$

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The problem:

Is there a polytime-computable relaxation R of the problem within a constant factor of the optimal?

Problem 5: Capacitated facility location

The problem:

Is there a polytime-computable relaxation R of the problem within a constant factor of the optimal?

Or, what's the approximate min-max relaxation?

$$R \leq \text{OPT} \leq \text{algorithm's soln} \leq \alpha R.$$

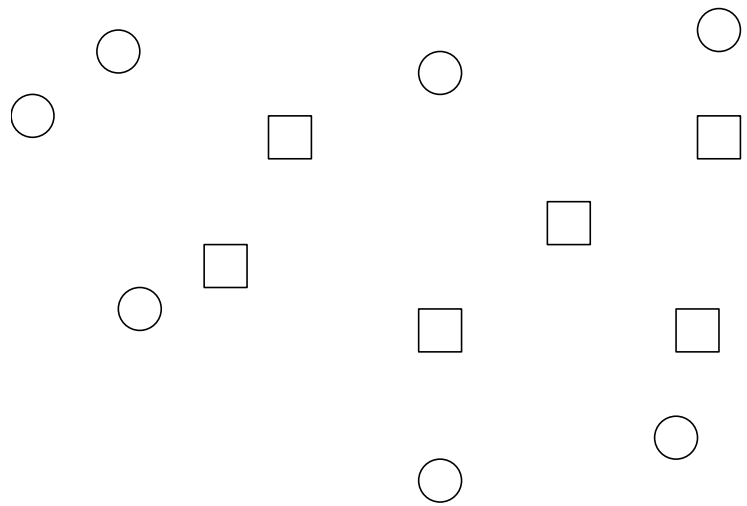
Problem 4: Survivable network design

Survivable network design

Input:

- An undirected graph $G = (V, E)$
- Costs $c_e \geq 0$ for all $e \in E$
- Integer connectivity requirements r_{ij} for all $i, j \in V$

Goal: Find a minimum-cost set of edges F so that for all $i, j \in V$, there are at least r_{ij} edge-disjoint paths between i and j in (V, F).



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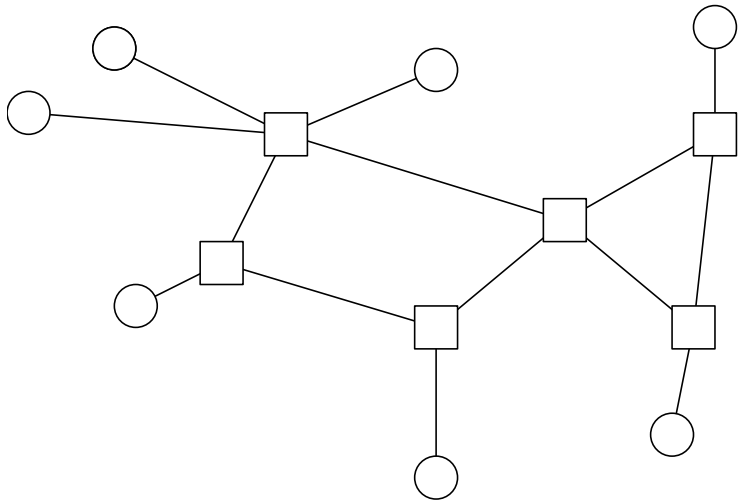

Problem 4: Survivable network design

What's known?

- A primal-dual $2H_R$-approximation algorithm (Goemans, Goldberg, Plotkin, Shmoys, Tardos, W '94), where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $R = \max_{i,j} r_{ij}$.
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- An LP rounding 2-approximation algorithm (Jain 2001)

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq \max_{i \in S, j \notin S} r_{ij}, \qquad \forall S \subset V, \\ & 0 \leq x_e \leq 1, \qquad \forall e \in E. \end{aligned}$$

Theorem (Jain 2001)

For any basic feasible solution x^* of the LP relaxation, there exists some edge $e \in E$ such that $x^*_e \geq 1/2$.

Problem 4: Survivable network design

The problem:

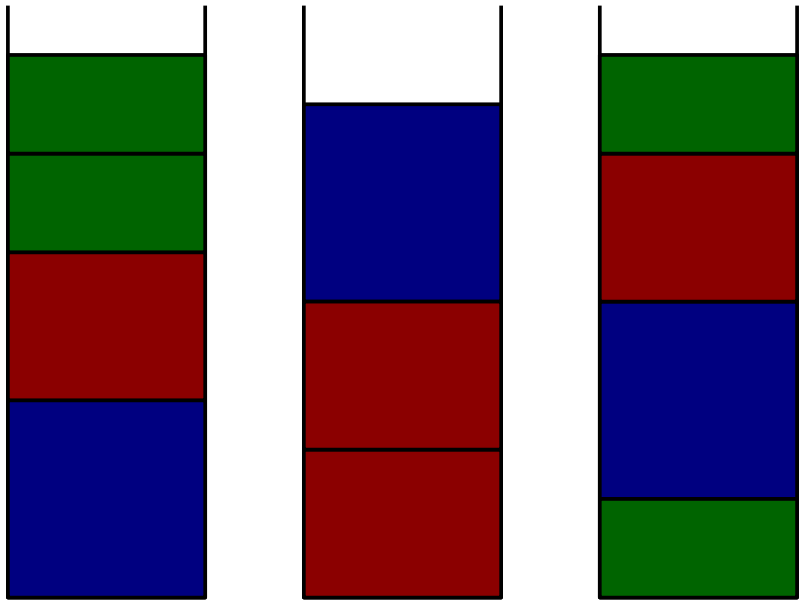
Is there a 2-approximation algorithm that doesn't require solving the LP? E.g. a primal-dual algorithm?

Problem 3: Bin packing

Bin packing

Input: b_i pieces of size s_i, $0 < s_i < 1$, for $i = 1, \ldots, m$

Goal: Find a packing of pieces into bins of size 1 that minimizes the total number of bins used



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An LP-rounding algorithm that uses $\mathrm{OPT} + O(\log^2 \mathrm{OPT})$ bins (Karmarkar, Karp 1982)

Enumerate all N possible ways of packing a bin. jth configuration uses a_{ij} pieces of size i.

$$\text{minimize} \quad \sum_{j=1}^{N} x_j$$

$$\text{subject to} \quad \sum_{j=1}^{N} a_{ij} x_j \geq b_i, \qquad i = 1, \ldots, m,$$

$$x_j \text{ integer}, \qquad j = 1, \ldots, N.$$

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Problem 3: Bin packing

The problem:

Find a polytime algorithm that uses at most OPT $+c$ bins for some constant c.

Note that there are instances known for which

$$\text{OPT} > \lceil LP \rceil + 1,$$

but currently no known instances for which

$$\text{OPT} > \lceil LP \rceil + 2.$$

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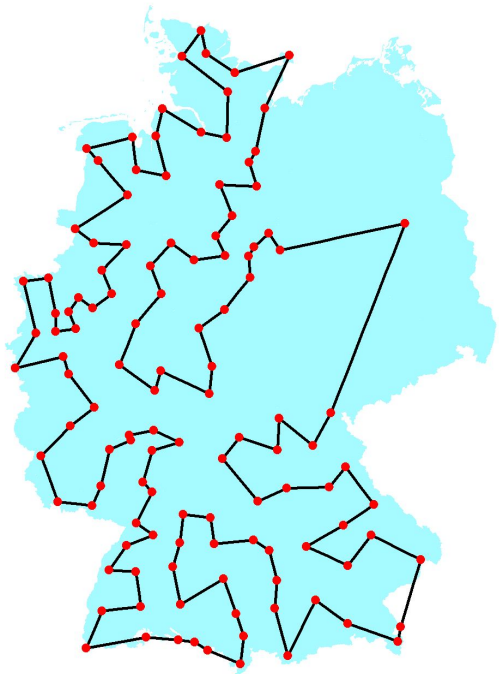
Problems 1 and 2: the traveling salesman problem

Traveling salesman problem

Input:

- Set of cities V
- Travel costs c_{ij} such that $c_{ij} \leq c_{ik} + c_{kj}$ for all $i, j, k \in V$

Goal: Find a minimum-cost tour of all the cities



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Problems 1 and 2: the traveling salesman problem

Problem 2: the asymmetric case ($c_{ij} \neq c_{ji}$)

What's known?

- An $O(\log n)$-approximation algorithm (Frieze, Galbiati, Maffioli 1982)
- An LP rounding $O(\log n / \log \log n)$-approximation algorithm (Asadpour, Goemans, Madry, Oveis Gharan, Saberi 2010)
- Can't approximate better than $\frac{117}{116} \approx 1.008$ unless P $=$ NP (Papadimitriou, Vempala 2006)

Problems 1 and 2: the traveling salesman problem

$$\text{minimize} \quad \sum_{i,j \in V} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j \in V} x_{ij} = \sum_{j \in V} x_{ji} \qquad i \in V,$$

$$\sum_{i \in S, j \notin S} x_{ij} \geq 1 \qquad \forall S \subset V$$

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The problem:

Find an α-approximation algorithm for α constant for the asymmetric case.

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Problems 1 and 2: the traveling salesman problem

Problem 1: the symmetric case $c_{ij} = c_{ji}$ for all $i, j \in V$

What's known?

- A $\frac{3}{2}$-approximation algorithm (Christofides 1976)
- Can't approximate better than $\frac{220}{219} \approx 1.004$ unless P = NP (Papadimitriou, Vempala 2006)

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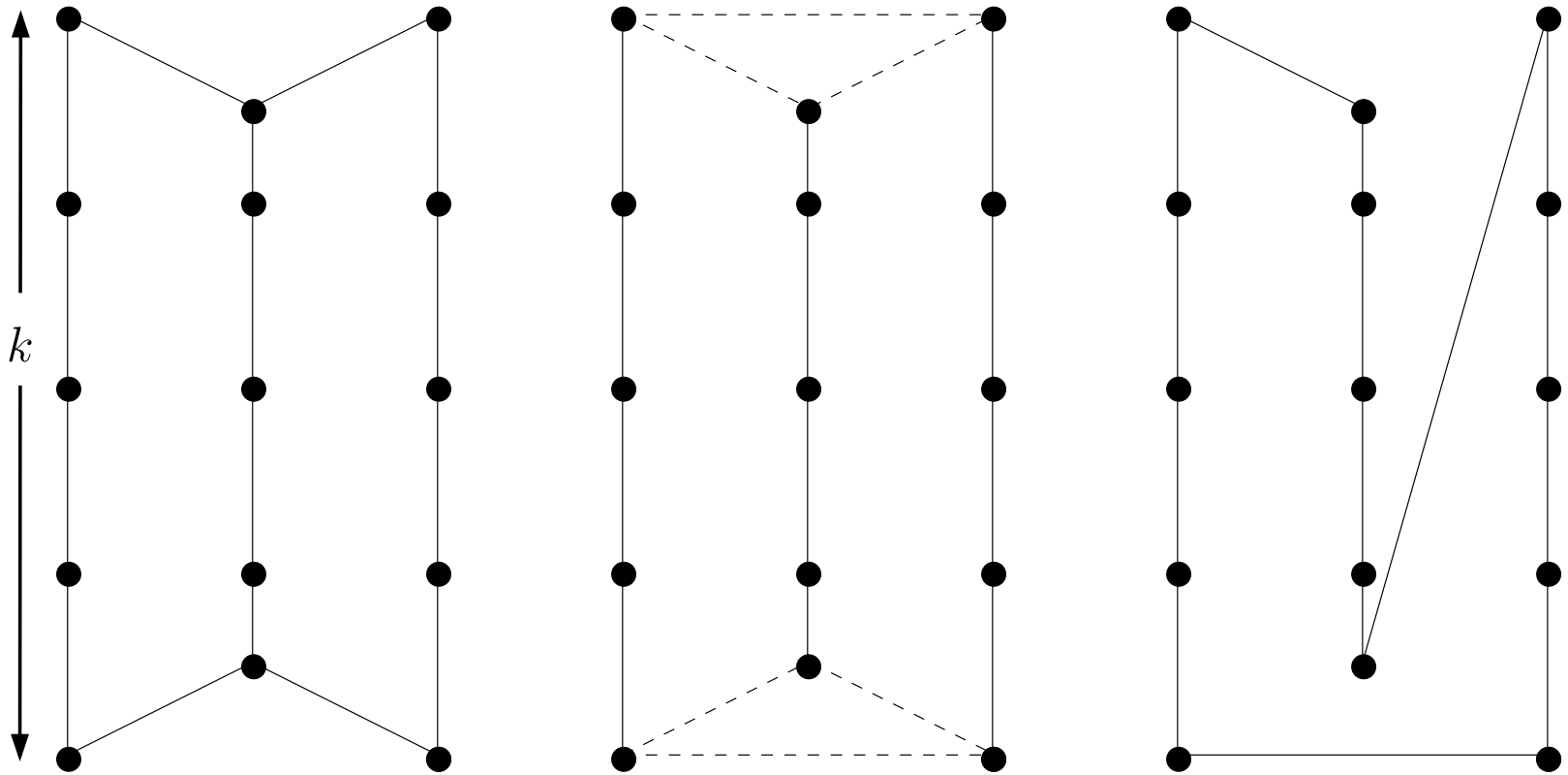
Problems 1 and 2: the traveling salesman problem

$$\begin{aligned}
\text{minimize} \quad & \sum_{i,j \in V : i<j} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{j \in V : i<j} x_{ij} + \sum_{j \in V : i>j} x_{ji} = 2 && i \in V \\
& \sum_{i \in S, j \notin S \text{ or } i \notin S, j \in S} x_{ij} \geq 2 && \forall S \subset V \\
& x_{ij} \geq 0 && \forall i, j \in V, i < j.
\end{aligned}$$

Integrality gap at most $\frac{3}{2}$ (Wolsey 1980). No instance known with gap worse than $\frac{4}{3}$.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Problems 1 and 2: the traveling salesman problem

The problem:

Find an α-approximation algorithm for constant $\alpha < \frac{3}{2}$.

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An observation

No open problem of the form "this problem has an α-approximation algorithm for constant α, find a PTAS."

Success in computation?

The field has certainly successfully generated interesting algorithmic ideas and mathematical understandings of approximate computation.

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But how much effect on actual computational practice?

Some cases in network design codes:

- Mihail, Shallcross, Dean, Mostrel (1996): Use primal-dual survivable network design algorithm
- Johnson, Minkoff, Phillips (2000): Use primal-dual prize-collecting Steiner tree algorithm

Also cases for problems that are theoretically solvable in polytime, but for which approximation algorithms are much faster: e.g. Müller, Radke, Vygen (2010)

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Can the theory help explain the realities of practice?

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Lightweight approximation algorithms

Perhaps part of the problem of adopting approximation algorithms is that the theoretically best algorithms are too computationally demanding compared to heuristics.

Examples:

- Best approximation algorithm for survivable network design requires solving LP via ellipsoid method
- Best approximation algorithm for max cut requires solving semidefinite program

Can we create *lightweight* approximation algorithms that deliver the same performance guarantees but with more practical computational requirements?

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And perhaps you!

The End

Thanks for your attention.

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